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### History of Physics (17)

Einstein in 1916: “On the Quantum Theory of Radiation”

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## Einstein in 1916: “On the Quantum Theory of Radiation”

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### Introduction

A peak in Einstein’s endeavor to extract as much information as possible about the nature of radiation from the Planck distribution is his paper “On the Quantum Theory of Radiation” of 1916 [1]<sup>1</sup>. In the first part he gives a derivation of Planck’s formula which has become part of many textbooks on quantum theory.

This part of the paper is now considered as the theoretical foundation of the laser, that was technically realized almost half a century later.

In the second part of his fundamental paper, Einstein discusses the exchange of momentum between atoms and radiation by making use of the theory of Brownian motion. Using a truly beautiful argument, which will be presented below, he shows that in every elementary process of radiation, and in particular in spontaneous emission, an amount  $h\nu/c$  of momentum is emitted in a random direction and that the atomic system suffers a corresponding recoil in the opposite direction.

This recoil was first experimentally confirmed in 1933 by showing that a long and narrow beam of excited sodium atoms widens up after spontaneous emissions have taken place [2]. Einstein’s paper [1] ends with the following remarkable statement concerning the role of “chance” in his description of the radiation processes by statistical laws, to which Pauli [3] drew special attention:

*“The weakness of the theory lies, on the one hand, in the fact that it does not bring us any closer to a merger with the undulatory theory, and, on the other hand, in the fact that it leaves the time and direction of elementary processes to ‘chance’; in spite of this I harbor full confidence in the trustworthiness of the path entered upon.”*

### Derivation of the Planck distribution

In his novel derivation of the Planck distribution, Einstein added the hitherto unknown process of induced emission<sup>2</sup>, next to the familiar processes of spontaneous emission and induced absorption. For each pair of energy levels he described the statistical laws for these processes by three coefficients (the famous  $A$ - and  $B$ -coefficients) and established two relations between these coefficients on the basis of his earlier correspondence argument in the classical Rayleigh-Jeans limit and Wien’s displacement law. In addition, the latter implies that the energy difference  $\varepsilon_n - \varepsilon_m$  between two internal energy states of the atoms in equilibrium with thermal radiation has to satisfy Bohr’s frequency condition:

$$\varepsilon_n - \varepsilon_m = h\nu_{nm}.$$

Einstein was very pleased by this derivation, about which he wrote on August 11, 1916 to Besso: “An amazingly simple

derivation of Planck’s formula, I should like to say *the derivation*” (CPAE, Vol. 8, Doc. 250).

In Dirac’s 1927 radiation theory these results follow – without any correspondence arguments – from first principles.

### Brownian motion of an atom in the thermodynamic radiation field

In the second part, which Einstein regarded in his paper as more important, he investigates the exchange of momentum between atoms and radiation by studying the Brownian motion of molecules in the thermodynamic radiation field. A molecule experiences thereby two types of forces:

- A systematic drag force  $Rv$ , where  $v$  is the velocity of the molecule, that leads in a small time interval  $(t, t + \tau)$  to the momentum change  $Rv\tau$ .
- An irregular change of momentum  $\Delta$  in the time  $\tau$ , due to fluctuations of the radiation pressure.

In thermal equilibrium the following equation holds:

$$\langle (Mv - Rv\tau + \Delta)^2 \rangle = \langle (Mv)^2 \rangle.$$

Assuming that  $\langle v \cdot \Delta \rangle = 0$  we obtain the important relation, repeatedly used by Einstein in his earlier work on Brownian motion,

$$\langle \Delta^2 \rangle = 2R\langle Mv^2 \rangle \tau = 2RkT\tau. \quad (1)$$

### Calculation of the friction $R$

The computation of  $R$  is not simple. For interested readers we present the details of Einstein’s instructive calculation, thereby including some flesh of his profound investigation. (Others may prefer to jump directly to the simple result, given in equation (3) below.)

The moving atom (molecule) sees an anisotropic radiation field. The friction force is the result of absorption and induced emission processes<sup>3</sup>. Let  $K$  be the rest system of the radiation field, with respect to which the radiation intensity is isotropic. With  $K'$  we denote the rest system of the atom. Let  $S$  denote the partition sum

$$S := g_n e^{-E_n/kT} + g_m e^{-E_m/kT} + \dots$$

for the energies  $E_n$  and degeneracies  $g_n$  of the atom. The fraction of time in the state  $n$  is equal to  $g_n e^{-E_n/kT} / S$ , and similarly for the state  $m$ . Therefore, the number of absorptions  $n \rightarrow m$  per unit time into the solid angle  $d\Omega'$  is equal to

$$\frac{1}{S} g_n e^{-E_n/kT} B_m^n \rho_{\nu_0}' \frac{d\Omega'}{4\pi}, \quad \nu_0 = \frac{E_m - E_n}{h},$$

where  $\rho_{\nu}$  denotes the energy density of the radiation as a function of frequency  $\nu$ . Correspondingly, for induced emission  $m \rightarrow n$  the number is

<sup>1</sup> The paper was first published in: *Physikalische Gesellschaft Zürich. Mitteilungen* 18 (1916): 47-62. The same paper was received 3 March and published 15 March 1917 in: *Physikalische Zeitschrift* 18 (1917): 121-128.

<sup>2</sup> Einstein’s derivation shows that without assuming a non-zero probability for induced emission one would necessarily arrive at Wien’s instead of Planck’s radiation law.

<sup>3</sup> The spontaneous emission has in the average no preferred direction, and therefore does in the average not transmit momentum to the atom.

$$\frac{1}{S} g_m e^{-E_m/kT} B_m^n \rho_{\nu_0}^1 \frac{d\Omega'}{4\pi}.$$

With Einstein's relation  $g_m B_m^n = g_n B_m^m$  from the first part of the paper, the momentum transfer per unit time in the x-direction is

$$-Rv = \frac{h\nu_0}{c} \frac{1}{S} g_n B_m^n (e^{-E_n/kT} - e^{-E_m/kT}) \times \int \rho_{\nu_0}^1(\theta', \varphi') \cos\theta' \frac{d\Omega'}{4\pi}.$$

### Calculation of $\rho_{\nu_0}(\theta', \varphi')$

With elementary calculations one can prove the following Lorentz invariance

$$\frac{\rho_{\nu'}}{\nu'^3} = \frac{\rho_\nu}{\nu^3}.$$

(This follows more directly from the fact that  $\rho(\nu)$  is proportional to  $\nu^3$  times the Lorentz invariant distribution function  $f$ .) Using also the Doppler shift to first order in  $v/c$ :  $\nu = \nu'(1 + \frac{v}{c} \cos\theta')$ . This gives

$$\rho_{\nu'} \simeq (1 + \frac{v}{c} \cos\theta')^{-3} \rho_{[1 + \frac{v}{c} \cos\theta']\nu'},$$

thus

$$\begin{aligned} \rho_{\nu_0}^1 &\simeq \left[ \rho_{\nu_0} + \left( \frac{\partial \rho_\nu}{\partial \nu} \right)_{\nu_0} \nu_0 \frac{v}{c} \cos\theta' \right] \left( 1 - 3 \frac{v}{c} \cos\theta' \right) \\ &\simeq \rho_{\nu_0} + \frac{v}{c} \cos\theta' \left[ \nu_0 \left( \frac{\partial \rho_\nu}{\partial \nu} \right)_{\nu_0} - 3\rho_{\nu_0} \right]. \end{aligned}$$

To first order in  $v/c$  we obtain

$$\begin{aligned} \int \rho_{\nu_0}^1 \cos\theta' \frac{d\Omega'}{4\pi} &= -\frac{v}{c} \left[ \rho_{\nu_0} - \frac{1}{3} \nu_0 \left( \frac{\partial \rho_\nu}{\partial \nu} \right)_{\nu_0} \right] \\ &= \frac{v}{c} \left[ \frac{\nu^4}{3} \frac{\partial}{\partial \nu} \left( \frac{\rho_\nu}{\nu^3} \right) \right]_{\nu=\nu_0}. \end{aligned}$$

Writing  $\nu$  instead of  $\nu_0$ , we arrive at

$$R = \frac{h\nu}{c^2} \frac{1}{S} g_n B_m^n e^{-E_n/kT} (1 - e^{-h\nu/kT}) \times \left[ -\frac{\nu^4}{3} \frac{\partial}{\partial \nu} \left( \frac{\rho_\nu}{\nu^3} \right) \right].$$

For the Planck distribution we obtain

$$R = \frac{1}{3} \left( \frac{h\nu}{c} \right)^2 \frac{1}{kT} \frac{g_n}{S} e^{-E_n/kT} B_m^n \rho_\nu. \quad (2)$$

Note that  $(g_n/S) e^{-E_n/kT} B_m^n \rho_\nu$  is the number of absorptions per unit time. Since this is equal to 1/2 times the number  $Z$  of elementary processes (induced and spontaneous emissions plus absorptions) per unit time, we find for the friction force the simple result:

$$R = \frac{1}{3} \left( \frac{h\nu}{c} \right)^2 \frac{1}{2kT} Z. \quad (3)$$

### Interpretation

Using this in (1) we arrive at the crucial relation

$$\langle \Delta^2 \rangle = \left( \frac{h\nu}{c} \right)^2 \langle \cos^2 \vartheta \rangle_{\mathcal{S}} Z \tau. \quad (4)$$

From classical electrodynamics (and experience) we know that to a pencil of light with energy  $\varepsilon$  belongs a momentum  $\varepsilon/c$  in the direction of the light beam. From this Einstein concluded that the atom receives in an absorption process the momentum  $(E_m - E_n)/c$  in the direction of the infalling radiation, and in an induced emission process the same momentum in the opposite direction. The crucial relation (4) then implies that *spontaneous* emission must also be directed in such a way that for every elementary process of radiation an amount  $h\nu/c$  of momentum is emitted in a random direction, and that the atomic system suffers a corresponding recoil in the opposite direction. "There is no radiation in spherical waves" <sup>4</sup>, Einstein adds.

Einstein regarded this conclusion as the main result of his paper. In a letter to Michele Besso he states:

*"With this, the existence of light-quanta is practically assured."* <sup>5</sup>

Two years later he added:

*"I do not doubt anymore the reality of radiation quanta, although I still stand quite alone in this conviction."*

We conclude with the following interesting remark of Einstein to O. Stern:

*"Ich habe hundertmal mehr über Quantenprobleme nachgedacht, als über die allgemeine Relativitätstheorie."*

### References

- [1] Einstein, A. On the Quantum Theory of Radiation, CPAE, *The collected papers of Albert Einstein*, Edited by J. Stachel et al., Vols. 1-12. Princeton: Princeton University Press, 1987-2010; Vol. 6, Doc. 38.
- [2] Frisch, R. (1933) Experimenteller Nachweis des Einsteinschen Strahlungsrückstoßes. *Zeitschrift für Physik*, 86, 42-48.
- [3] Pauli, W. (1949) Einstein's contributions to Quantum Theory. In *Albert Einstein: Philosopher-Scientist*, edited by P. A. Schilpp. Illinois: The Library of Living Philosophers.

Noch eine allgemeine Bemerkung soll hier Platz finden. Fast alle Theorien der Temperaturstrahlung beruhen auf der Betrachtung der Wechselwirkungen zwischen Strahlung und Molekülen. Aber im allgemeinen begnügt man sich mit der Betrachtung des Energie-Austausches, ohne den Impuls-Austausch zu berücksichtigen. Man fühlt sich dazu leicht berechtigt, weil die Kleinheit der durch die Strahlung übertragenen Impulse es mit sich bringt, daß letztere gegenüber anderen bewegungserzeugenden Ursachen in der Wirklichkeit fast stets zurücktreten. Aber für die theoretische Betrachtung sind jene kleinen Wirkungen neben den ins Auge fallenden der Energieübertragung durch Strahlung als durchaus gleichwertig anzusehen, indem Energie und Impuls aufs engste miteinander verknüpft sind; es kann deshalb eine Theorie erst dann als berechtigt angesehen werden, wenn gezeigt ist, daß die nach ihr von der Strahlung auf die Materie übertragenen Impulse zu solchen Bewegungen führen, wie sie die Wärmetheorie verlangt.

*The last paragraph from Einstein's original paper (from [1]).*

<sup>4</sup> "Ausstrahlung in Kugelwellen gibt es nicht".

<sup>5</sup> "Damit sind die Lichtquanten so gut wie gesichert" (6. September 1916).