Iterieren & Diskretisieren

Mathematische Schlüsselkonzepte für die digitale Welt

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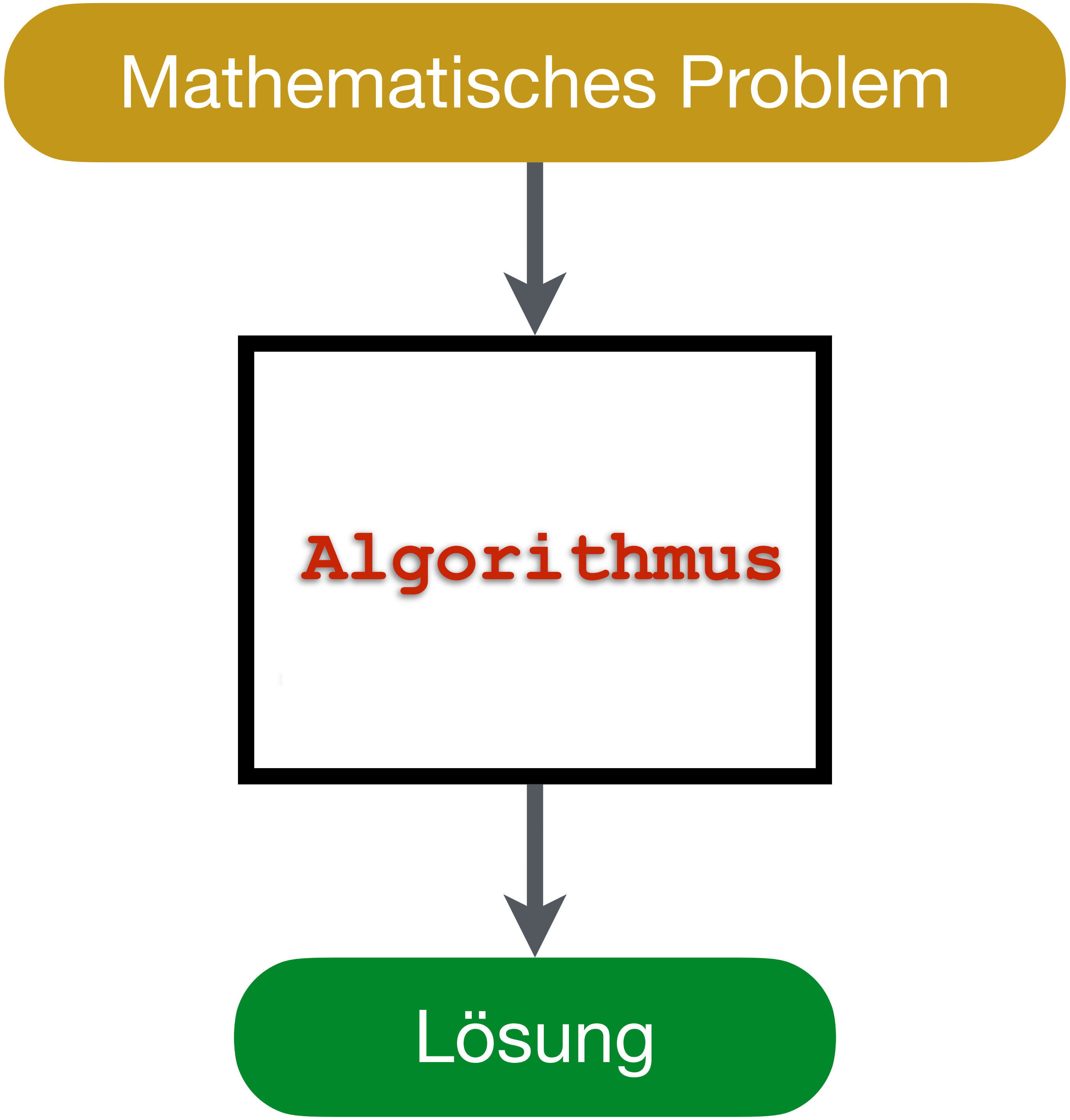
~~Numerical analysis is the study of rounding errors~~

—L. N. Trefethen

Numerical analysis is the study of algorithms for the problems of continuous mathematics


Mathematisches Problem
Algorithmus
Lösung

Mathematisches Problem

endlicher Speicher

endliche Prozesse

Diskretisieren

Iterieren

Lösung

Näherung

Qualität?

“Symbole”

$$\int_0^1 \frac{2x+1}{x^2+1}\,\mathrm{d}x = \frac{\pi}{4} + \ln(2) \approx 1.47855$$

Zuverlässigkeit

$$x^2 - 2x + 1 = 10^{-20} \quad\Longrightarrow\quad x = 1 \pm 10^{-10}$$

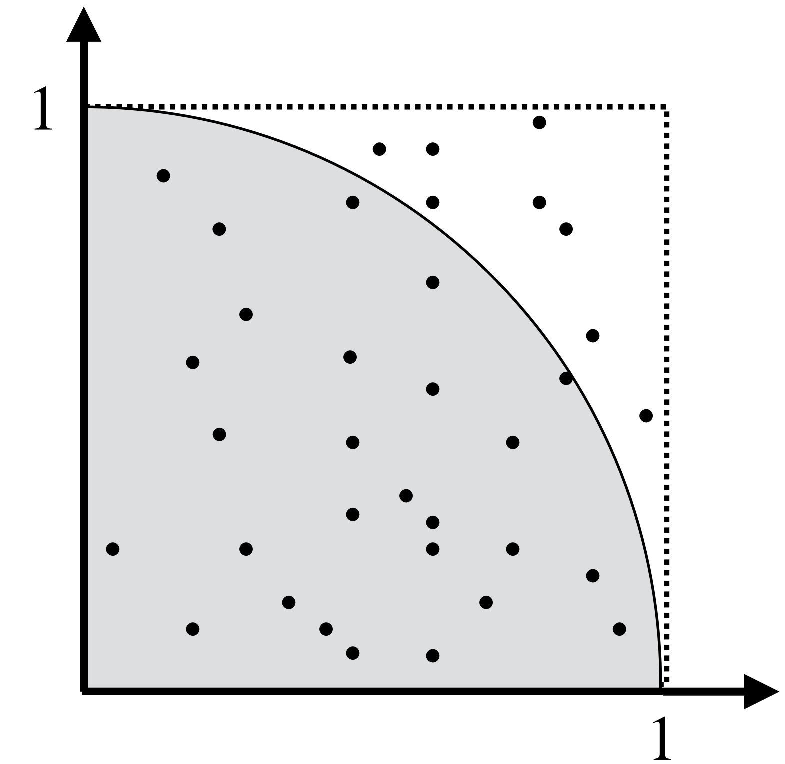
zwei *einfache* Nullstellen
exakt in $\mathbb{M}$ darstellbar

$$\Big\downarrow \mathbb{M}$$

$$x^2 - 2x + 1 = 0 \quad\Longrightarrow\quad x = 1$$

doppelte Nullstelle

Effizienz



- Binomialverteilung mit

$$p = \mathbb{P}[\text{Erfolg}] = \pi/4$$

- $\mathbb{E}[n] = pn$ Treffer aus total n Würfen

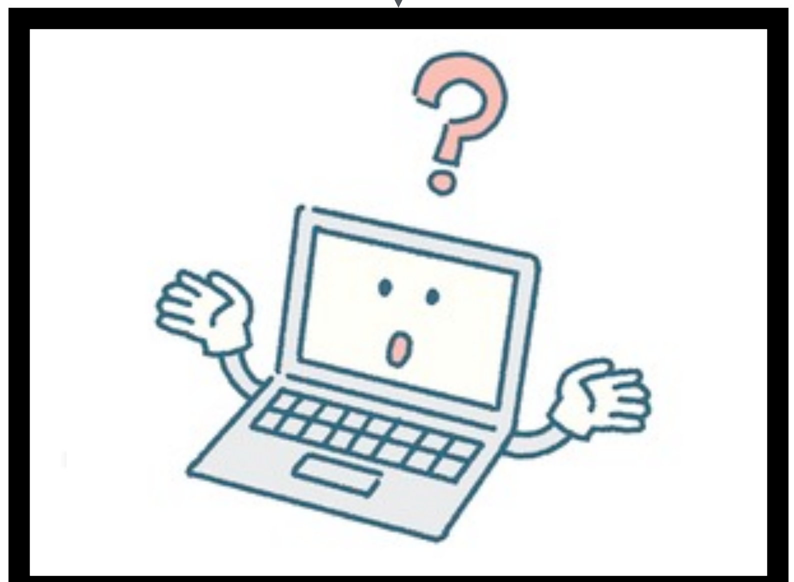
- Beste Approximation der Fläche:

$$A_n = \frac{\mathrm{rnd}(pn)}{n} \xrightarrow{n\to\infty} p = \pi/4$$

Funktionsauswertung

$\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \ldots$

$\mathbb{M}$



$+\,,\,-\,,\,\times\,,\,\div$

$=\,,\,\neq\,,\,<\,,\,\ldots$

$\exp, \ln$

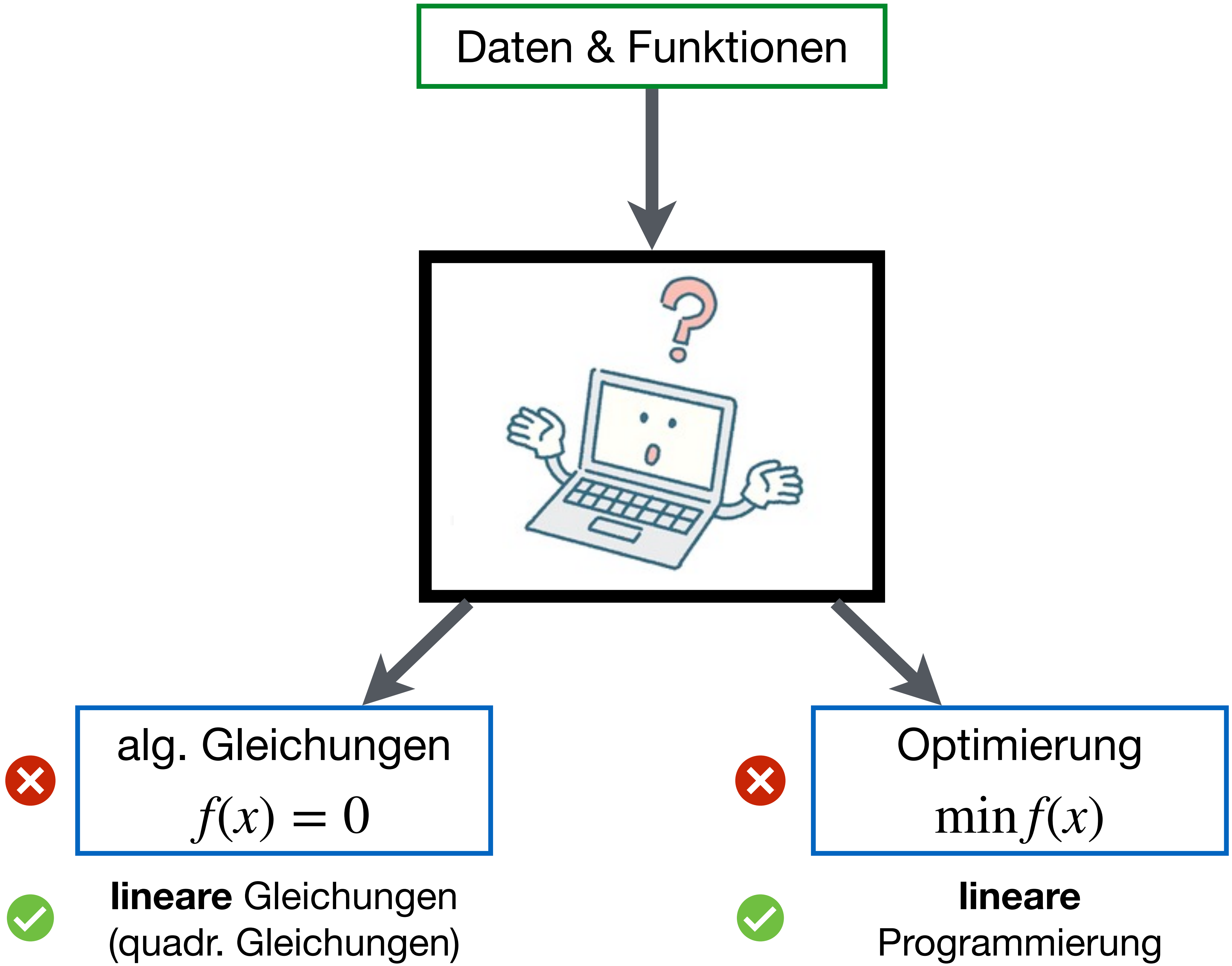
$x^p, \quad \sin, \cos, \ldots$

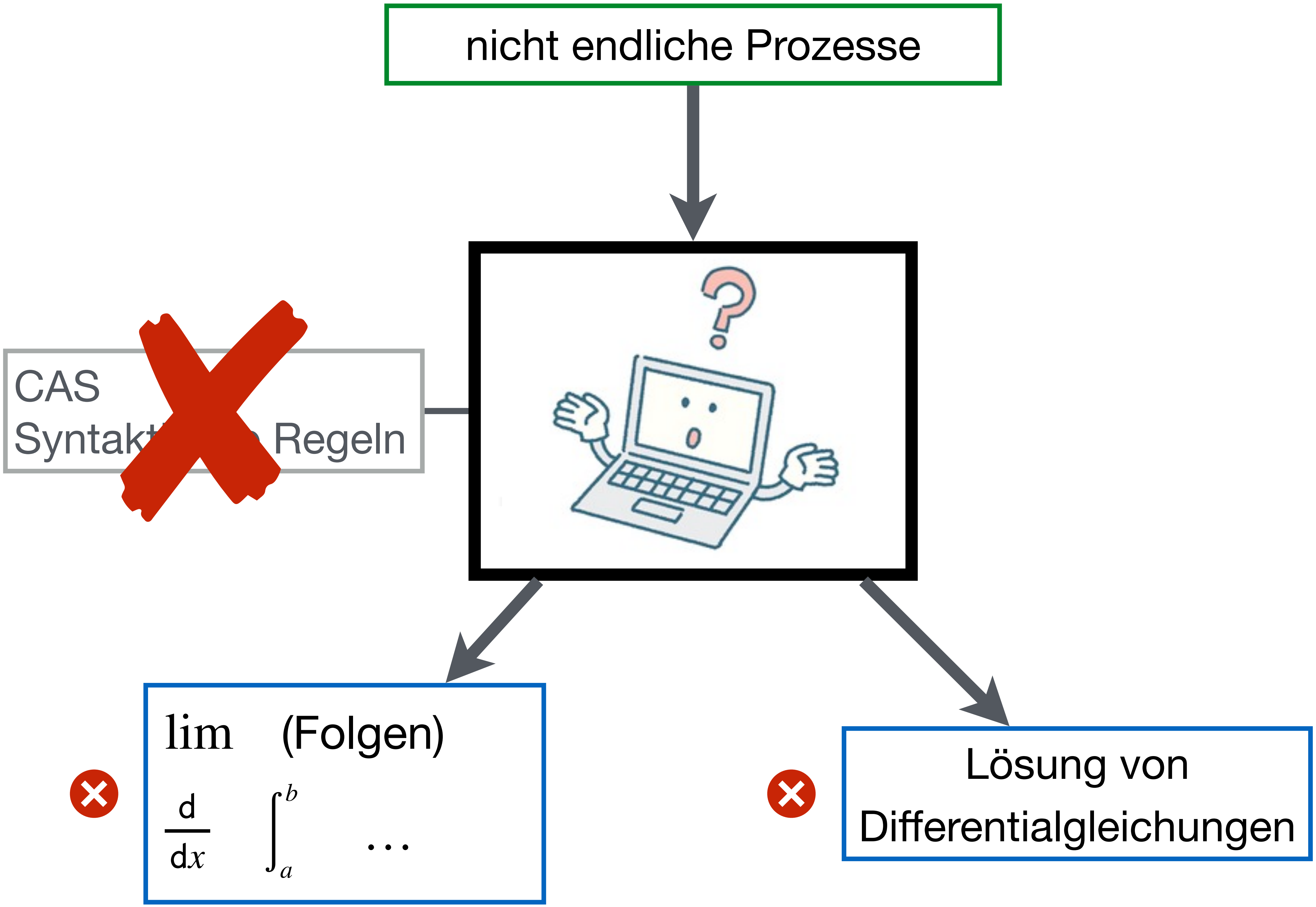
gute Approximationen
der Elementarfunktionen

(stetige) Funktionen

$x \mapsto f(x)$

$\mathbb{M}$-Polynome
kein ZWS


Daten & Funktionen
alg. Gleichungen
$f(x) = 0$
Optimierung
$\min f(x)$
lineare Gleichungen
(quadr. Gleichungen)
lineare
Programmierung


nicht endliche Prozesse
CAS
Syntakt
Regeln
lim (Folgen)
$\frac{\mathrm{d}}{\mathrm{d}x}$ $\int_a^b$ $\dots$
Lösung von Differentialgleichungen

Mathematik **praktisch** am Ende?

Diskretisieren

Iterieren

endlich dimensionale, lineare & iterative Strukturen

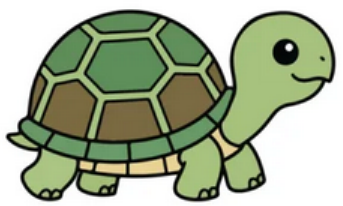
numerische Verfahren

“Sprache” der **linearen Algebra**

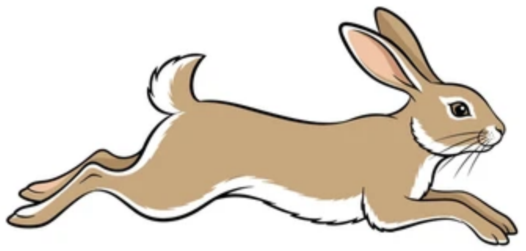
Endliche Approximationen

Exponentialfunktion

$$e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n \approx \left(1+\frac{x}{N}\right)^N, \qquad N \gg 1$$



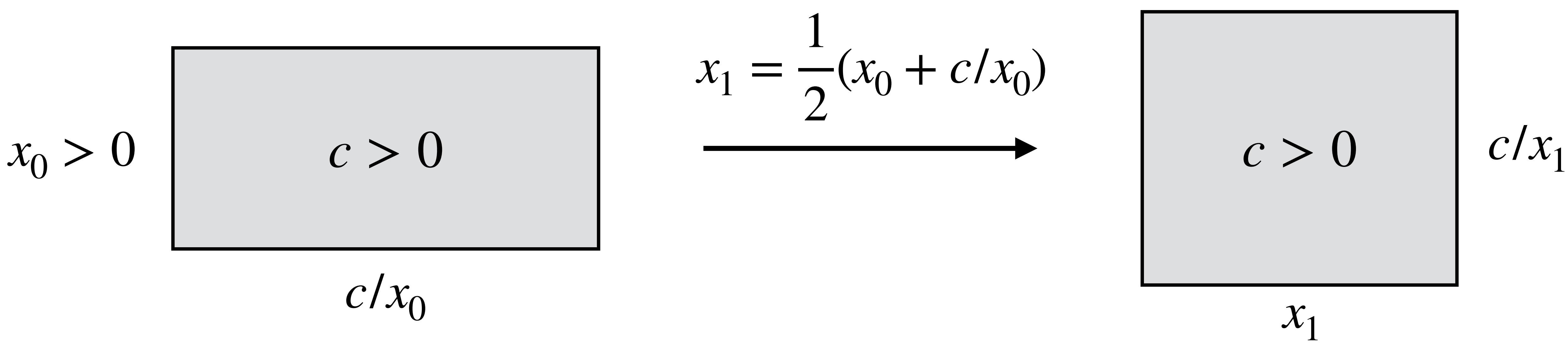
$$e^x = \sum_{k=0}^{\infty}\frac{x^k}{k!} \approx \sum_{k=0}^{N}\frac{x^k}{k!} =: S_N(x), \qquad N > 1, \quad x \approx 0$$



$$e^x = \left(e^{x/m}\right)^m \approx S_N(x/m)^m \qquad m = 2^k, \quad x/m \approx 0$$

Iteration

$\sqrt{c} = ?$



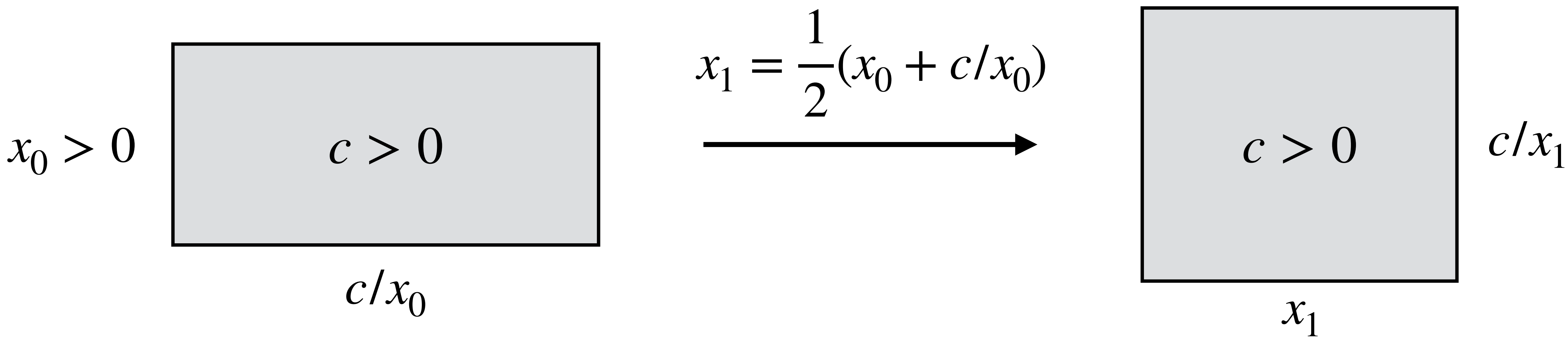
Heron-Verfahren:

$$\boxed{x_{n+1} = \frac{1}{2}\left(x_n + \frac{c}{x_n}\right), \qquad n \geq 0}$$

Satz:

Für alle $x_0 > 0$ gilt $x_n \to \sqrt{c}$ mit quadratischer Konvergenz.

$\sqrt{c} = ?$



Heron-Verfahren:

$$\boxed{x_{n+1} = \frac{1}{2}\left(x_n + \frac{c}{x_n}\right), \qquad n \geq 0}$$

Beweis:

$x_{n+1} \geq \sqrt{c}$ (AG-Ungleichung)

$$x_{n+1} = \tfrac{1}{2}x_n\left(1 + \underbrace{c/x_n^2}_{\leq 1}\right) \leq x_n$$

A posteriori Fehler-Analysis

$$\left|\sqrt{c} - x_n\right| = \frac{\left|\left(\sqrt{c} - x_n\right)\left(\sqrt{c} + x_n\right)\right|}{\sqrt{c} + x_n}$$

$$= \frac{|c - x_n^2|}{\sqrt{c} + x_n}$$

$$= \frac{x_n^2 - c}{\sqrt{c} + x_n} \qquad x_n \geq \sqrt{c}$$

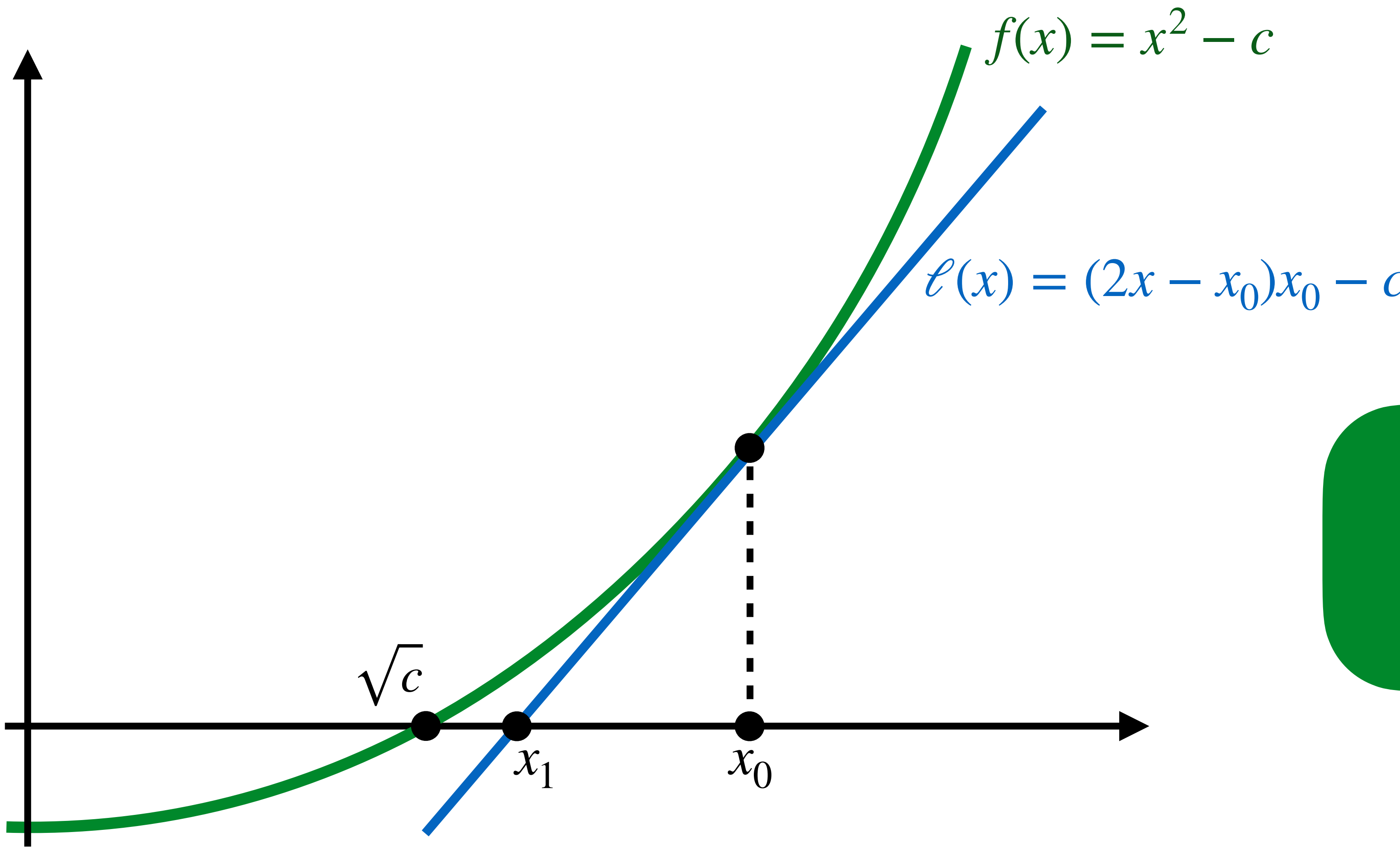
$$\leq \frac{x_n^2 - c}{c/x_n + x_n} \qquad \sqrt{c} = c/\sqrt{c} \geq c/x_n$$

A posteriori Fehler-Analysis

n	true error	*a posteriori* upper bound
1	$0.857864 \cdot 10^{-1}$	$0.882353 \cdot 10^{-1}$
2	$0.245310 \cdot 10^{-2}$	$0.245523 \cdot 10^{-2}$
3	$0.212390 \cdot 10^{-5}$	$0.212390 \cdot 10^{-5}$
4	$0.159472 \cdot 10^{-11}$	$0.159474 \cdot 10^{-11}$

Approximation von $\sqrt{2}$

Iteration durch Linearisierung



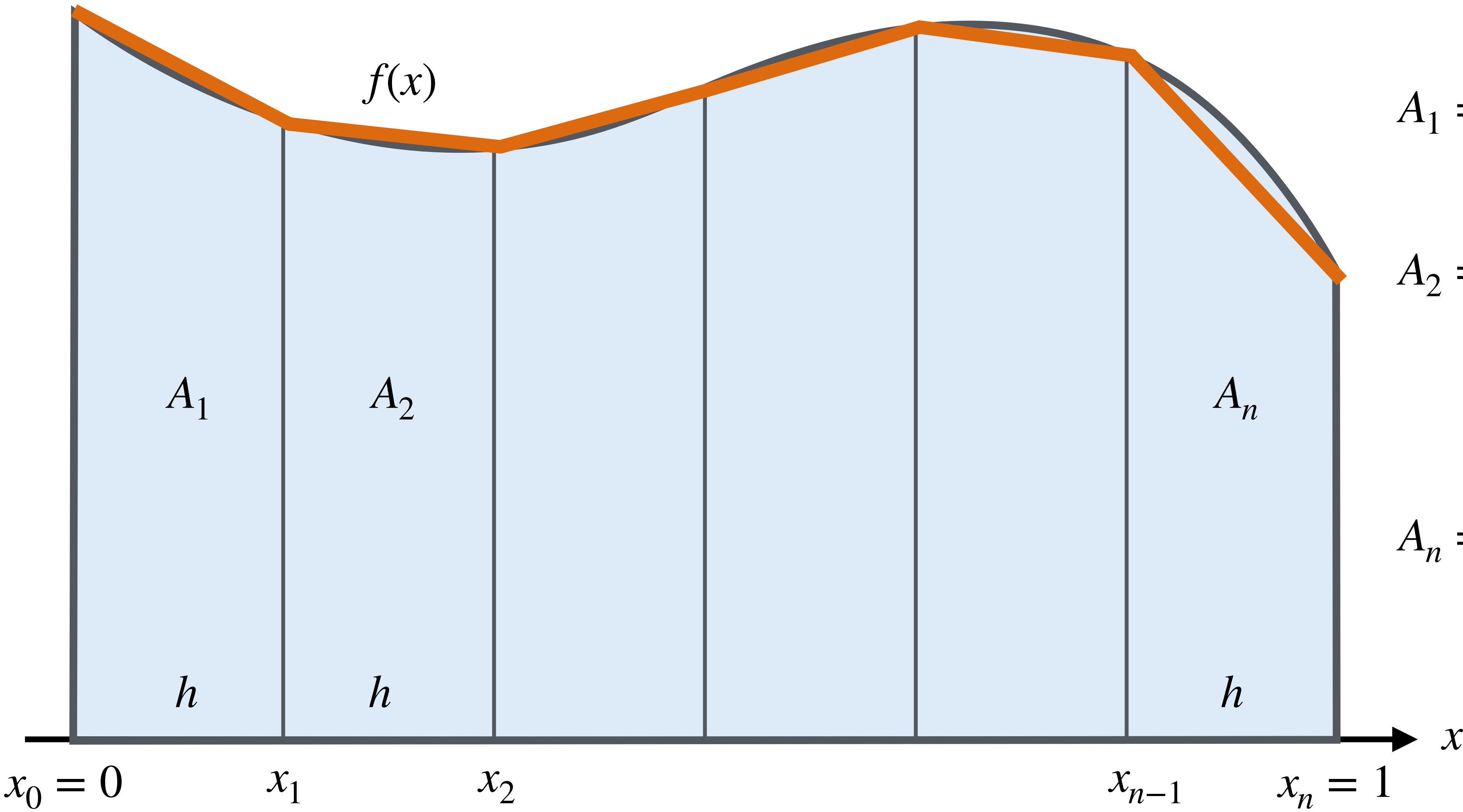
Iteration durch Linearisierung

Heron-Verfahren:

$$\boxed{x_{n+1} = \frac{1}{2}\left(x_n + \frac{c}{x_n}\right), \qquad n \geq 0}$$

Newton-Verfahren: Ersetze nichtlineares Problem durch eine Folge von **linearen** Problemen.

Diskretisierung



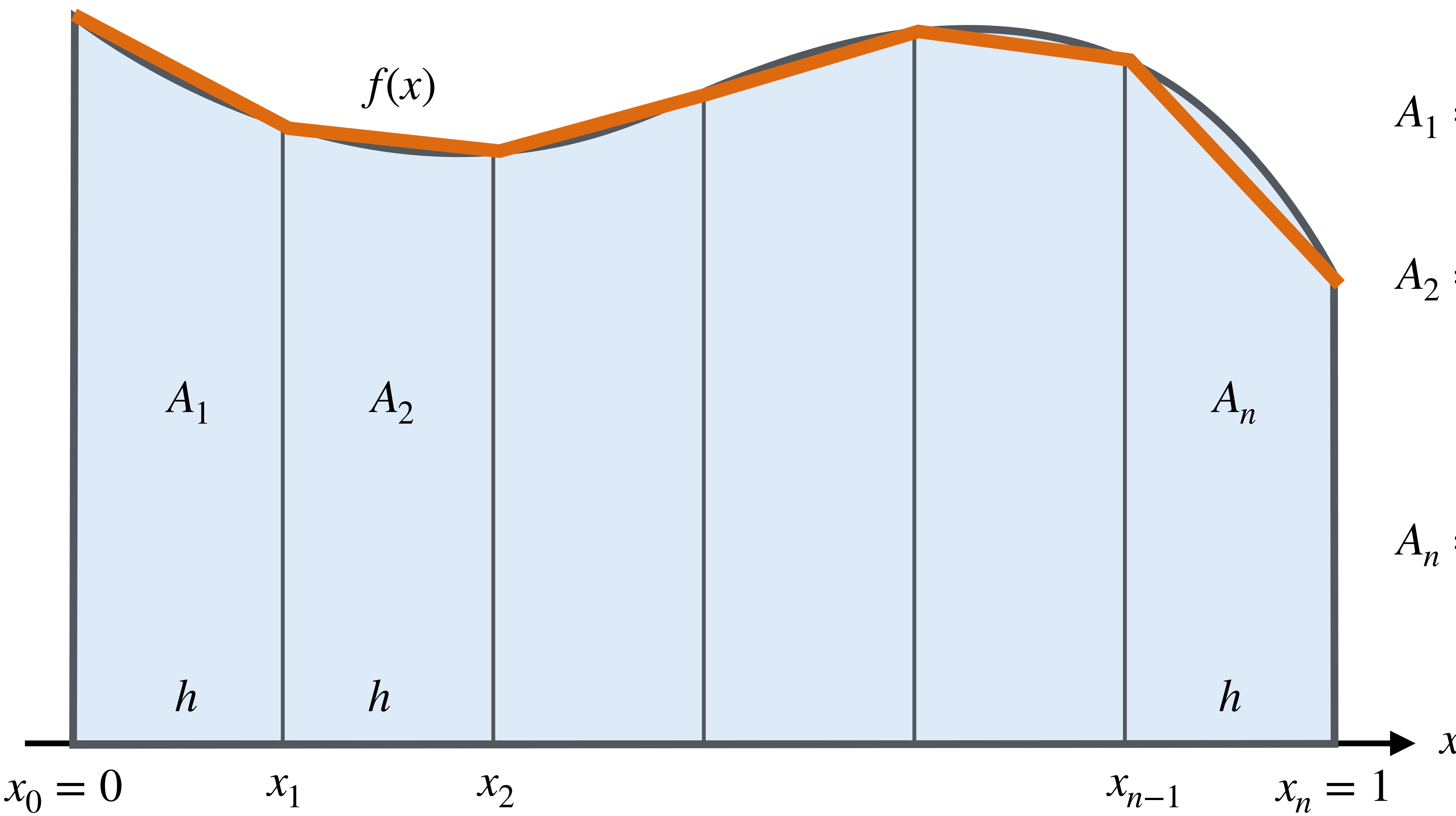
$$A_1 = \frac{h}{2}(f(x_0) + f(x_1))$$

$$A_2 = \frac{h}{2}(f(x_1) + f(x_2))$$

$$\vdots$$

$$A_n = \frac{h}{2}(f(x_{n-1}) + f(x_n))$$

$$\int_0^1 f(x)\,\mathrm{d}x \approx A_1 + \cdots + A_n = \frac{h}{2}\big(f(x_0) + 2f(x_1)) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\big)$$



$$A_1 = \frac{h}{2}\left(f(x_0) + f(x_1)\right)$$

$$A_2 = \frac{h}{2}\left(f(x_1) + f(x_2)\right)$$

$$\vdots$$

$$A_n = \frac{h}{2}\left(f(x_{n-1}) + f(x_n)\right)$$

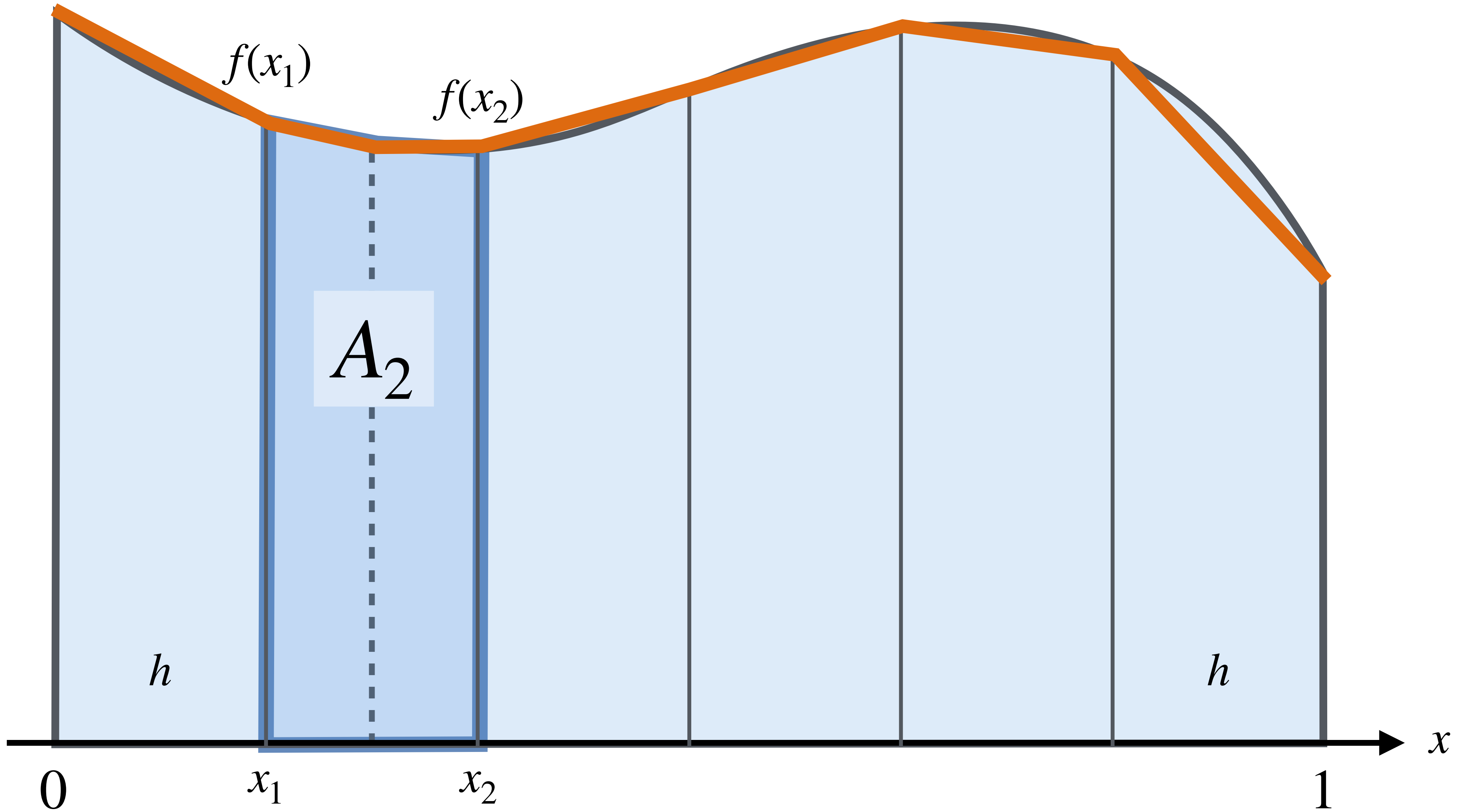
$$\int_0^1 f(x)\,\mathrm{d}x \approx \frac{h}{2}\begin{pmatrix} 1 \\ 2 \\ \vdots \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_{n-1}) \\ f(x_n) \end{pmatrix}$$

Trapezregel

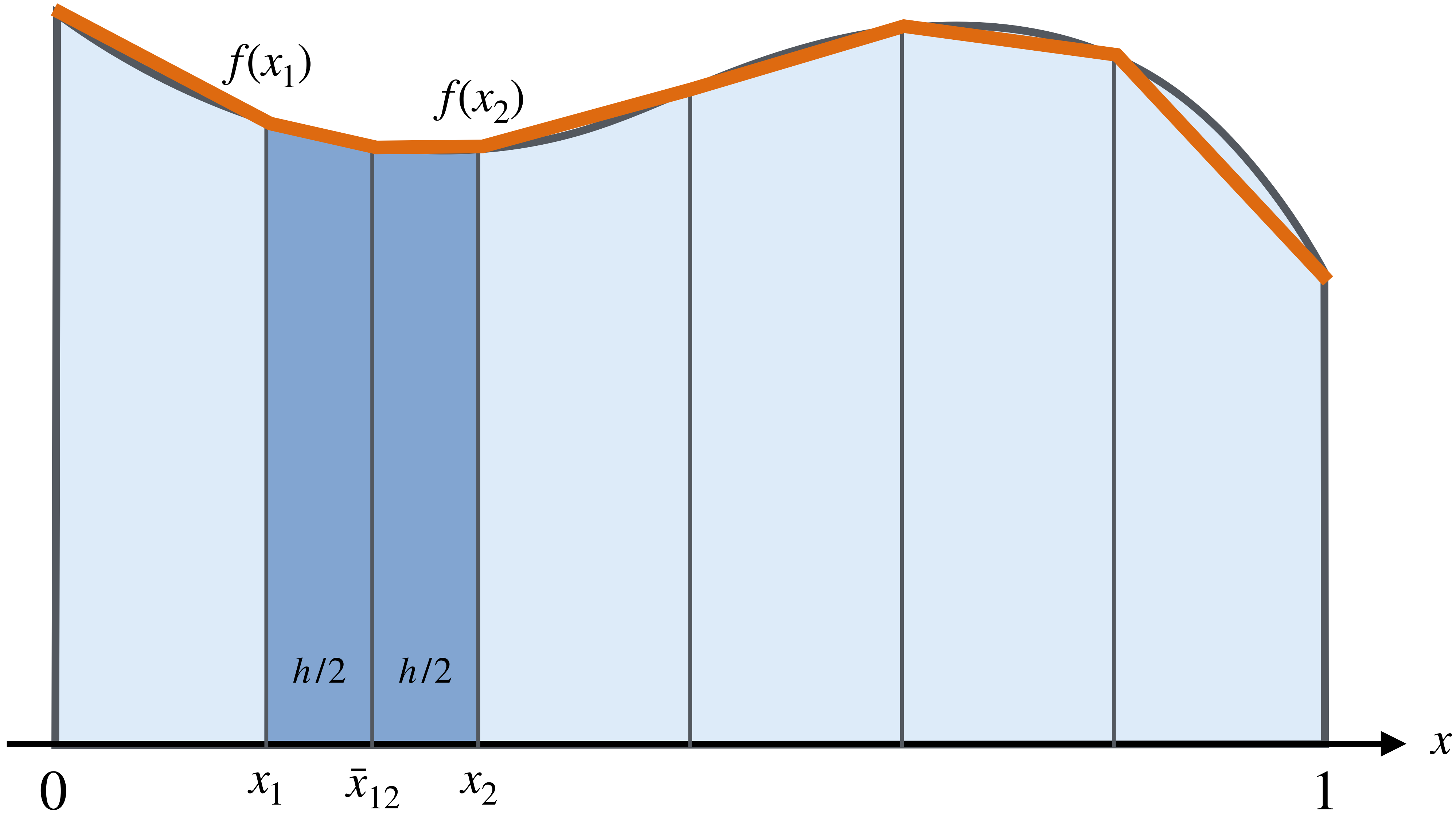
```
function A = TrapezRegel(f,a,b,n)

h = (b-a)/n;
x = linspace(a,b,n+1);
w = 2*ones(1,n+1); w([1 end]) = 1;
A = h/2*dot(f(x),w);

end
```

```
>> f=@(x) x.^2;
>> TrapezRegel(f,0,1,100)

ans =

   0.333350000000000
```

Iterative
Diskretisierung



$$A_2 = \frac{h}{2}(f(x_1) + f(x_2))$$



$$A_2 = \frac{h}{2}(f(x_1) + f(x_2))$$

$$\tilde{A}_2 = \frac{h}{4}(f(x_1) + 2f(\bar{x}_{12}) + f(x_2))$$

$$\Delta A_2 = |A_2 - \tilde{A}_2| = \underbrace{\frac{h}{4}|f(x_1) - 2f(\bar{x}_{12}) + f(x_2)|}_{\text{\textit{a posteriori} Indikator (Krümmung)}}$$

Adaptive Integration

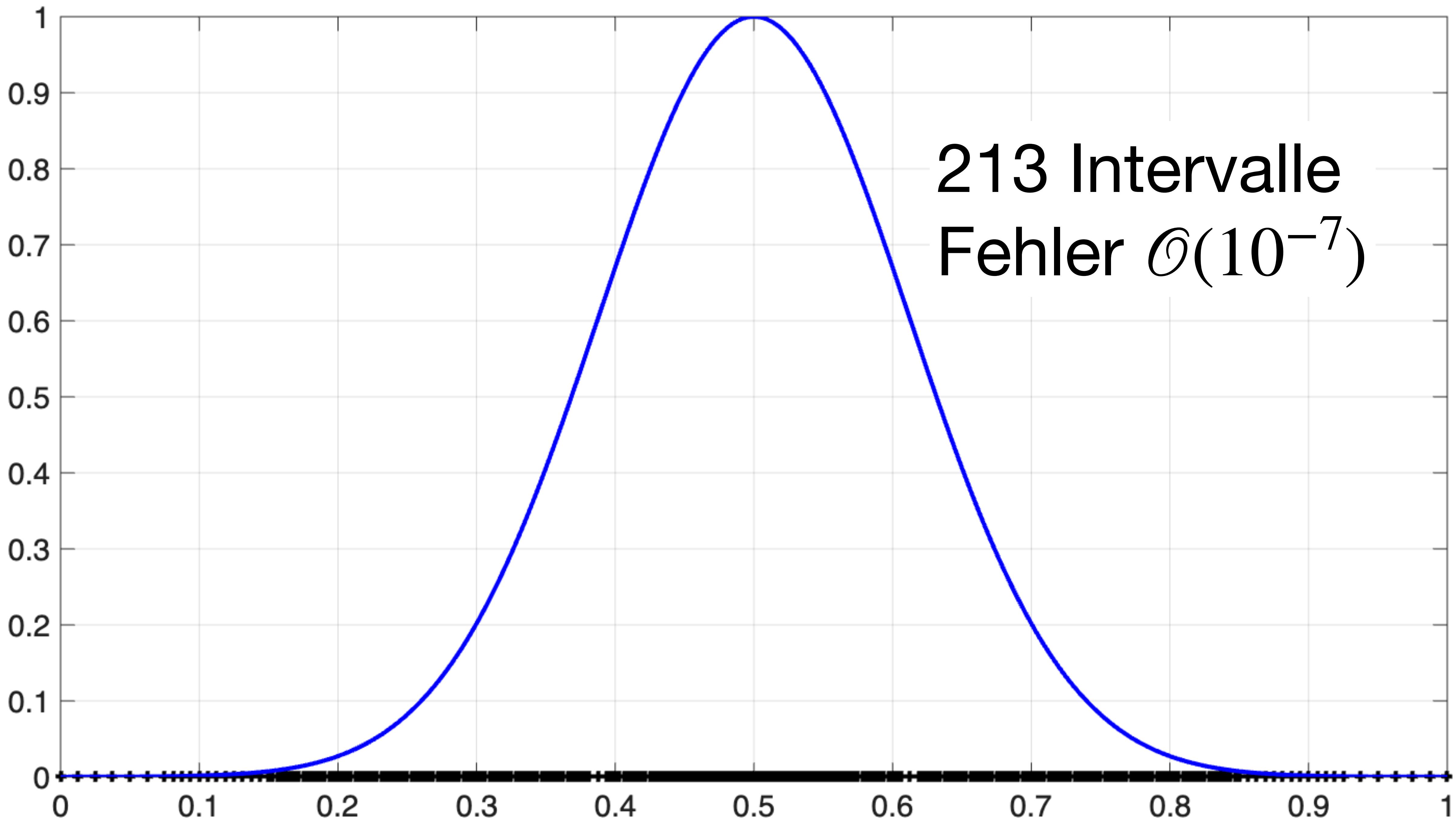
- Unterteile das Integrationsgebiet in n Intervalle der Länge h und berechne die Trapezflächen unter dem Graphen von f
- **Iteriere {**
  - Unterteile jedes Intervall in zwei Teilintervalle der Länge $h/2$ und berechne jeweils die entsprechenden Trapezflächen
  - Bestimme die a posteriori Indikatoren $\Delta A_1, \ldots, \Delta A_n$
  - Identifiziere und unterteile **adaptiv** alle Intervalle k, für die

$$\Delta A_k \geq \alpha \max_{1 \leq j \leq n} \Delta A_j$$

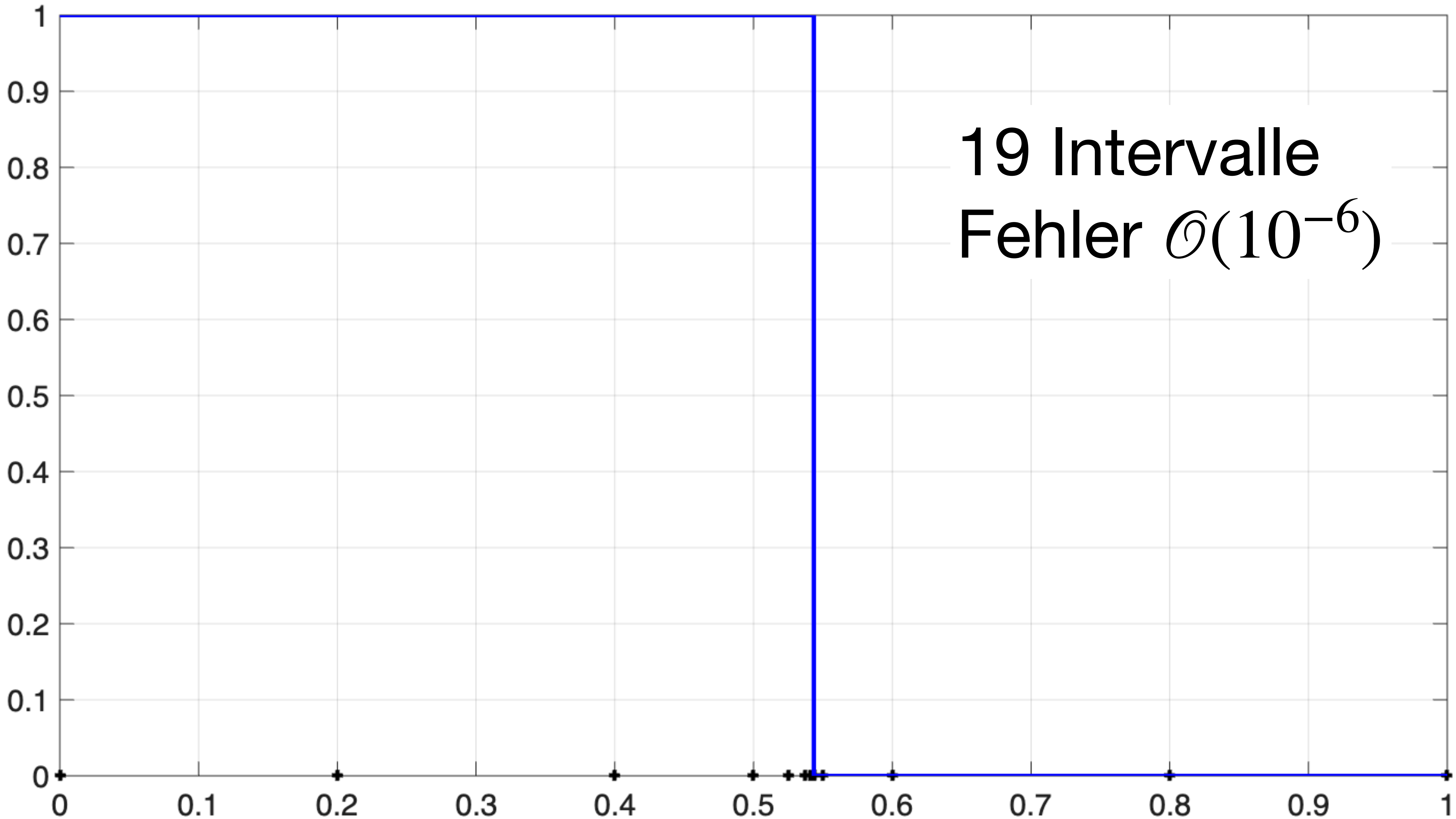
  gilt, für $0 < \alpha < 1$.

  }
- Stoppe, sobald die globale Verbesserung vernachlässigbar ist

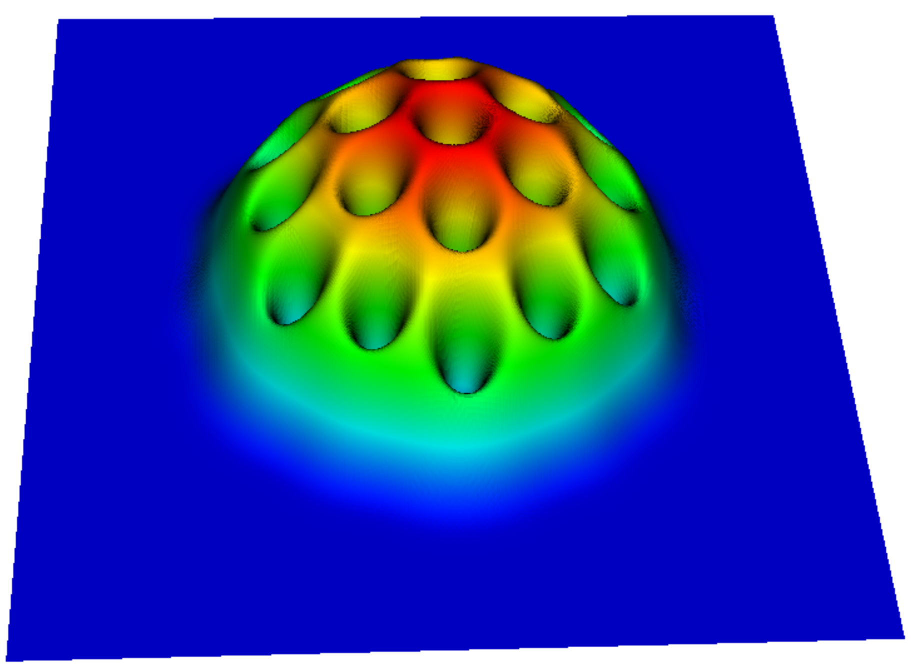
$$\int_0^1 \exp\left(-40(x-0.5)^2\right)\, \mathrm{d}x \approx 0.27922837\ldots$$



$$\int_0^1 \chi_{x<0.5434}\,\mathrm{d}x = 0.5434$$



State of the art



$$E(u) := \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + V(\mathbf{x})|u|^2 + \frac{\beta}{2}|u|^4 - i\omega\overline{u}\,(\mathbf{A}(\mathbf{x})\cdot\nabla u)\right)\,\mathrm{d}\mathbf{x},$$

MATHEMATICS OF COMPUTATION

AMERICAN MATHEMATICAL SOCIETY

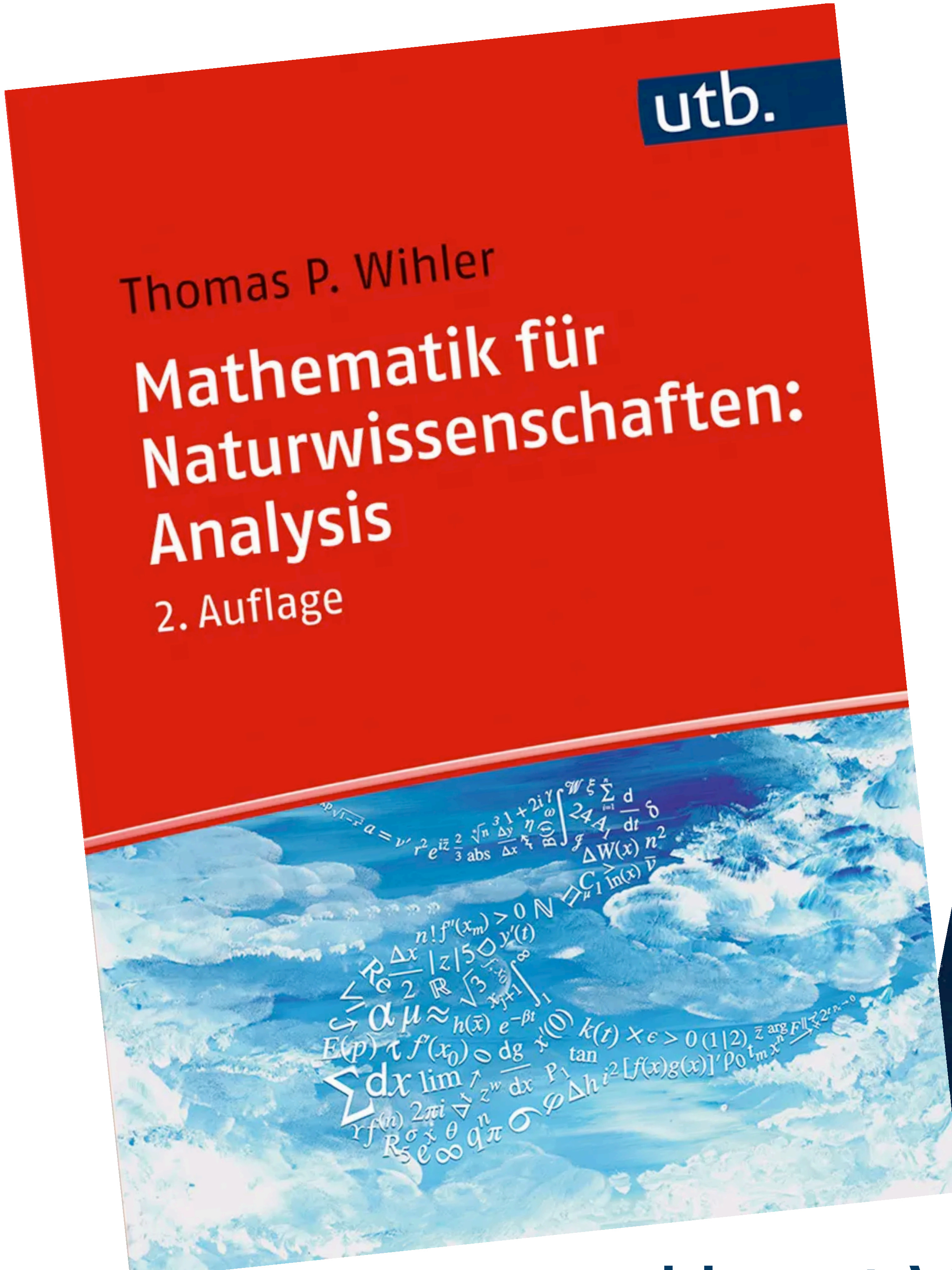
Adaptive iterative linearization Galerkin methods for nonlinear problems

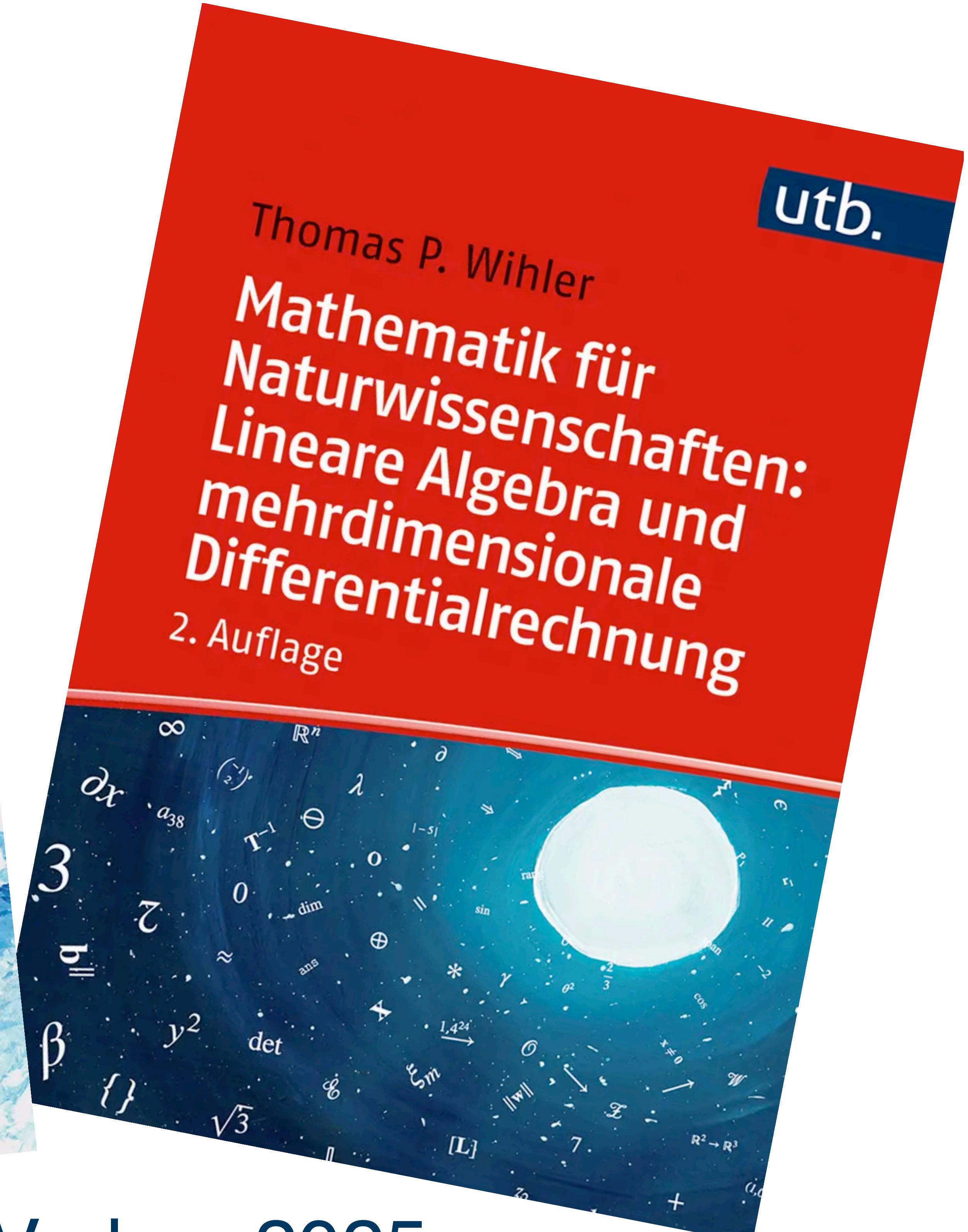
Pascal Heid and Thomas P. Wihler

Abstract

A wide variety of (fixed-point) iterative methods for the solution of nonlinear equations (in Hilbert spaces) exists. In many cases, such schemes can be interpreted as iterative *local linearization* methods, which, as will be shown, can be obtained by applying a suitable preconditioning operator to the original (nonlinear) equation. Based on this observation, we will derive a unified abstract framework which recovers some prominent iterative schemes. In particular, for Lipschitz continuous and strongly monotone operators, we derive a general convergence analysis. Furthermore, in the context of numerical solution schemes for nonlinear partial differential equations, we propose a combination of the iterative linearization approach and the classical Galerkin discretization method, thereby giving rise to the so-called *iterative linearization Galerkin (ILG)* methodology. Moreover, still on an abstract level, based on two different elliptic reconstruction techniques, we derive a posteriori error estimates which separately take into account the discretization and linearization errors. Furthermore, we propose an adaptive algorithm, which provides an efficient interplay between these two effects. In addition, the ILG approach will be applied to the specific context of finite element discretizations of quasilinear elliptic equations, and some numerical experiments will be performed.

Numerische Mathematik befasst sich mit der
Entwicklung und mathematischen Analyse
von ***effizienten Lösungsverfahren***
für mathematische Probleme,
die auf dem ***Rechner realisierbar*** *sind*
und deren ***Verlässlichkeit*** *durch*
fundierte theoretische Grundlagen *gesichert ist.*





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