Mathematical Modeling in Life Sciences Introduction

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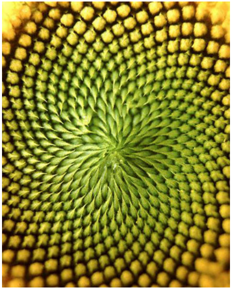
Phyllotaxis = Observation of regular pattern in the arrangement of leaves on a stem.

(a) Aeonium

(b) Cone



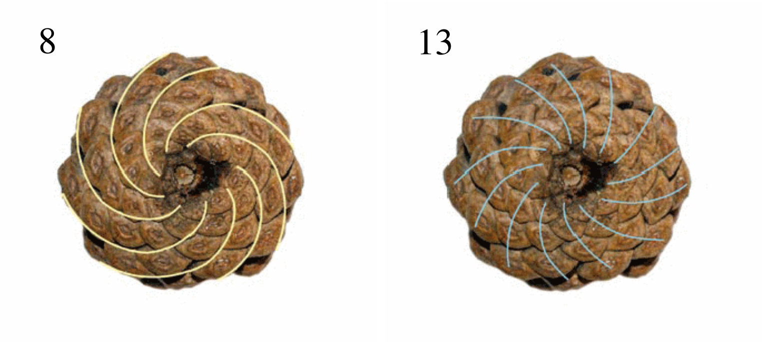
(c) Sunflower

Fibonacci numbers and plants

Fibonacci numbers : $X_0 = 1,\ X_1 = 1,\ X_{k+1} = X_k + X_{k-1}$
$\Longrightarrow 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$

2 and 3 are two consecutive Fibonacci numbers.

Fibonacci numbers and plants II



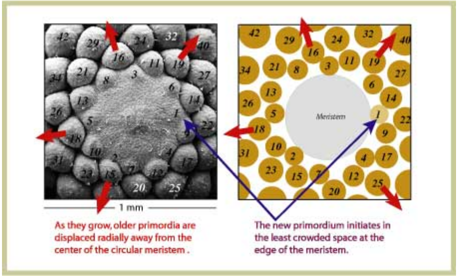
8 and 13 are also two consecutive Fibonacci's numbers.

Golden angle and plants

Figure: The angle between two consecutive leaves is $\pm$ constant: this is the **divergence angle**, which is well approximated by $\phi \approx 137.5$ deg.

Mathematical models I

- Hofmeister (1868) observed that the new primordium formed at the least crowded spot.



- Turing A. M. (1952) developed reaction-diffusion models to explain pattern. These models are used in morphogenesis in various settings, but do not seem to be relevant for phyllotaxis.

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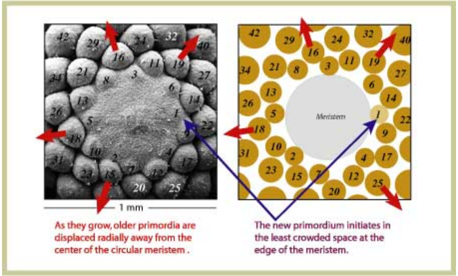

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- Adler I. (1974) used the idea of repulsion in a first model.
- Levitov L. (1991) vortex model, using repulsive potentials. Theory based on hyperbolic geometry, and numerical simulations which show the emergence of Fibonacci numbers and of the Golden Angle.
- Douady S. and Couder Y. (1992) proposed an experiment for the Levitov's model. Emergence of Fibonacci spirals.
- Kunz M. (1995). Rigorous mathematical study of these experiments using statistical mechanics. Relations with number theory.
- Atela P., Golé C. and Hotton S. (2002) Rigorous mathematical study of Adler's model, in the spirit of Levitov and Kunz. Strong use of hyperbolic geometry and number theory.

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- Smith et al. (2006). Model of active auxin transport, with PIN proteins.
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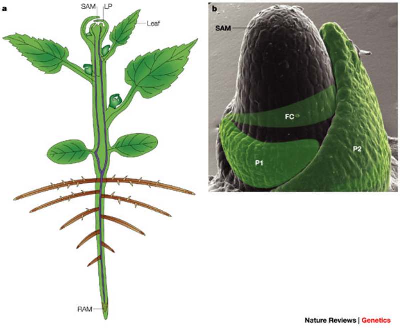
Anatomy of the plant



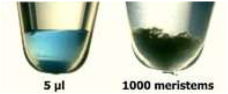
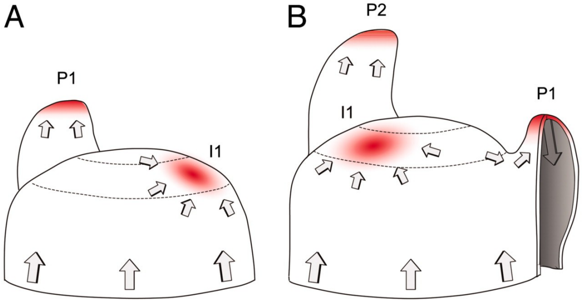


Figure: from M. Tsiantis and A. Hay, Nature Reviews Genetics 4, 2003

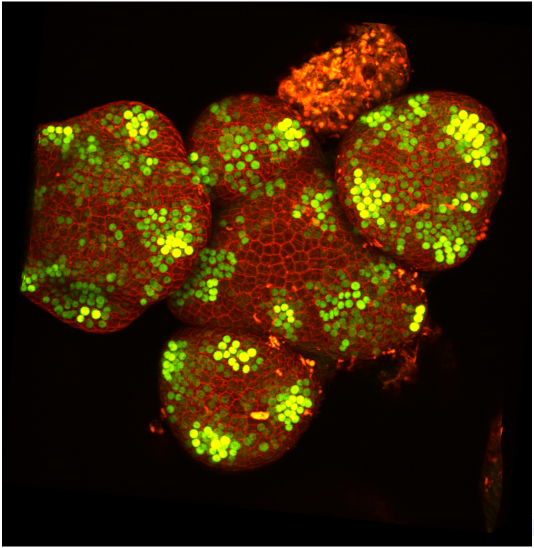
- Organ of a plant = leaves, flowers
- Initiation of the organ on the top of the stem (meristem).

Conceptual model of the regulation of phyllotaxis by polar auxin fluxes in the shoot meristem



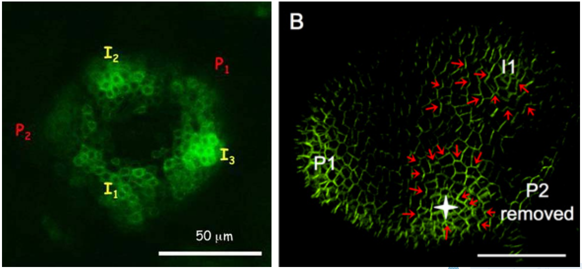
Smith R S et al. PNAS 2006;103:1301-1306

Regular patterns emerge in the meristem, as a consequence of auxin peaks formation



Zoom on the two biochemical processes

- Auxin can diffuse through cell membranes, but can also be transported with the help of PIN proteins.
- The PIN proteins are exporters of auxin $\Longrightarrow$ Polarization of the process
- This process creates an **auxin depleted zone around the incipient primordium.**



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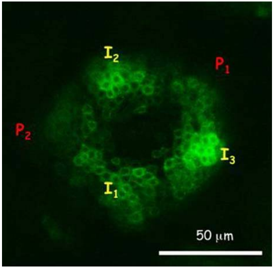
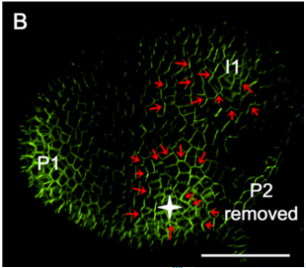
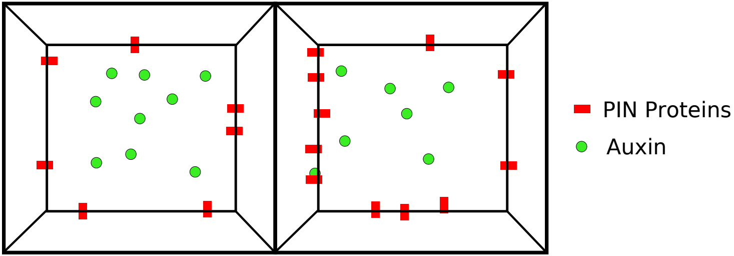
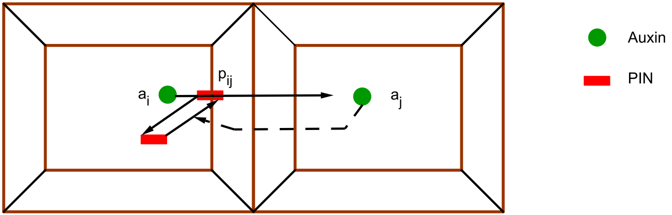




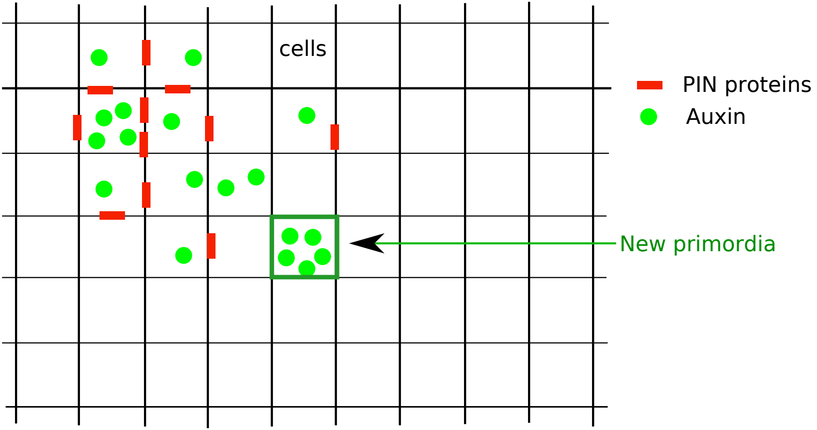
Schematic representation



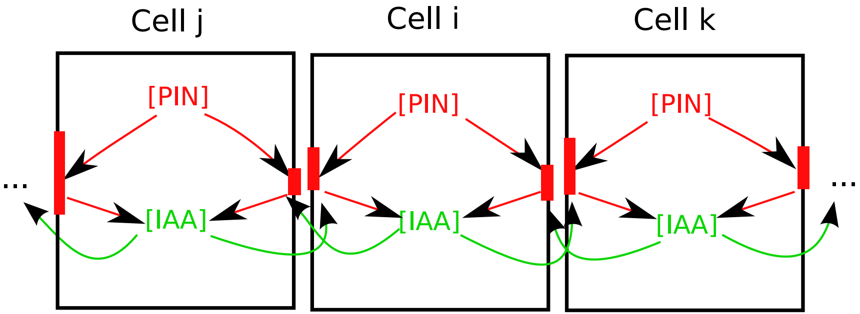
Concentration based model



PIN polarization, auxin peaks formation



$$
\begin{aligned}
a_i &= \text{auxin concentration in cell } i \text{ in mol}/m^3 \\
p_i &= \text{PIN concentration in cell } i \\
p_{ij} &= \text{PIN concentration on the membrane of cell } i \text{ facing cell } j \qquad (1)
\end{aligned}
$$



Model based on ordinary differential equations (o.d.e.)

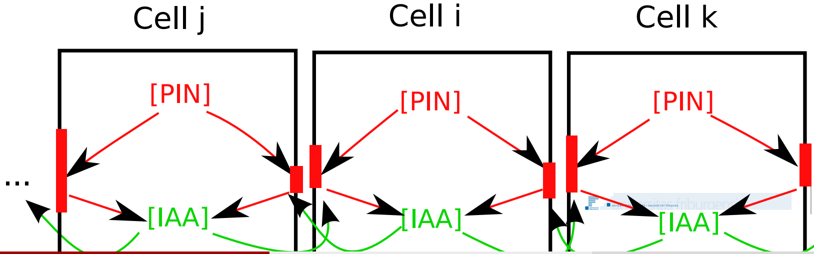
Equations of Jönsson et al. (2006):

$$\frac{da_i}{dt} = \mu - \nu a_i + D\sum_{k\sim i}(a_k - a_i) + T\sum_{k\sim i}(a_k p_{ki} - a_i p_{ik})$$

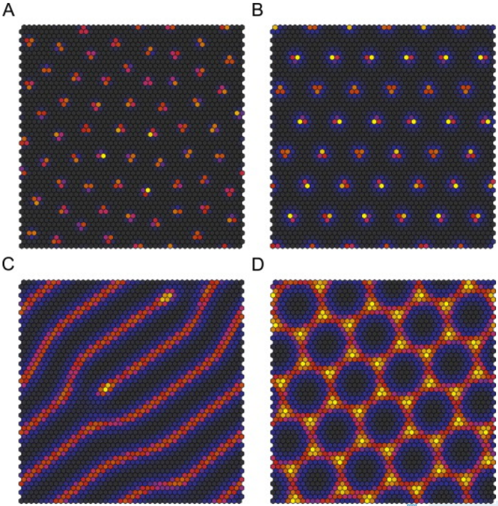
$$\frac{dp_{ij}}{dt} = f(a_j)p_i - k_2 p_{ij}$$

$$\frac{dp_i}{dt} = \sum_{k\sim i}(k_2 p_{ik} - f(a_k)p_i) \tag{2}$$

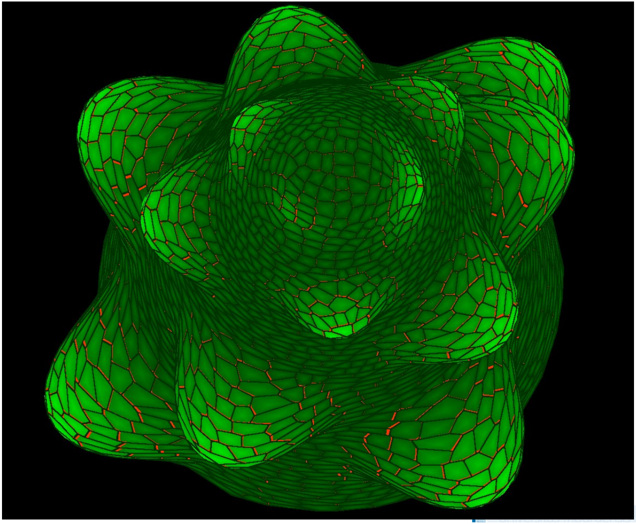
$$f(x) = k_1 x. \tag{3}$$



Simulation from Sahlin et al. (2009)



Simulation from Smith et al. (2009)



Modelling topics

- Pattern formation on growing domains using reaction-diffusion equations (Dagmar Iber)
- Models for photosynthesis and metabolic networks basd on o.d.e (Olivier Ebenhoe)
- Foodwebs modelling using graph theory (Louis-Félix Bersier)
- Stability of Lotka Volterra dynamical systems (Rudolf Rohr)
- Chemical reaction networks and Markov chains (Christian Mazza)

Problem sessions and group works are proposed for each of these topics. The projects involve simulations using either the statistical software R or MatLab

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