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1 Introduction

In a recent article we described how Wolfgang Pauli arrived at the exclusion principle hundred years ago, before the advent of quantum mechanics. One of the immediate great qualitative successes of quantum mechanics was that it implies the stability of atoms. A much less obvious consequence of the theory is that ordinary matter in bulk, held together by Coulomb forces, is also stable.

The mystery of this fact before the dawn of quantum mechanics was described by Jeans in 1915 with the following words [1]:

> *"There would be a very real difficulty in supposing that the (force) law $1/r^2$ held down to zero values of r. For the force between two charges at zero distance would be infinite; we should have charges of opposite sign continually rushing together and, when once together, no force would be adequate to separate them (...). Thus matter in the universe would tend to shrink into nothing or to diminish indefinitely in size."*

In quantum mechanics the electrons cannot fall into the nuclei.

We begin this article with the stability and size of atoms and then to the much less obvious consequence of the stability of matter in bulk. Four decades after non-relativistic quantum mechanics was developed, Dyson and Lenard gave the first rigorous proof of the stability for matter in bulk [2]. For this the Pauli principle for the electrons is essential, while the statistics of the nuclei does not matter. This was the beginning of a lot of remarkable work by several authors, in particular by F. J. Dyson, A. Lenard, E. H. Lieb, W. Thirring, and others.

Atoms and 'ordinary' matter in bulk, consisting of a system of N electrons and k nuclei with charges $Z_1e,...,Z_ke$, can be well described by the mutual Coulomb interactions. For the discussion that follows we use the Hamiltonian

$$H = T_e + V_{eK} + V_{ee} + V_{KK} \qquad (1).$$

T_e is the kinetic energy of the electrons, and the three potential energies V_{eK}, V_{ee}, V_{KK} are the Coulomb energies between the electrons and nuclei, among the electrons and among the nuclei, respectively. We treat the nuclei as infinitely heavy in fixed positions $\mathbf{R}_1, ..., \mathbf{R}_k$ (Born-Oppenheimer approximation). Since we are mainly interested in lower bounds of the ground state energy of the system, this is not a serious simplification; if the nuclei are treated dynamically, the nuclear kinetic energy adds a positive contribution.

Two different notions of stability are useful.
(i) *Stability of the first kind*:

$$E(N,k,\underline{R}) := inf_{\psi}\,(\psi, H\psi)$$

is finite for every N, k, and fixed positions $\underline{R} = (\mathbf{R}_1, ..., \mathbf{R}_k)$ of the nuclei.

(ii) *Stability of the second kind*: Assuming that $Z_j \le Z$ for all $j = 1, ..., k$, then

$$E(N,k) := inf_{\underline{R}}\, E(N,k,\underline{R}) \ge -A(Z)(N+k) \;,$$

where A depends only on Z.

2 Stability of atoms

The stability of the first kind for isolated atoms is obvious, even without the Pauli principle for electrons. The remarkable fact is that the exclusion principle guarantees stability of the second kind. This can easily be demonstrated.
For a single atom the Hamiltonian is

$$H_N = \frac{1}{2m}\sum_{i=1}^{N}\boldsymbol{p}_i^2 - Ze^2\sum_{i=1}^{N}\frac{1}{|\boldsymbol{x}_i|} + e^2\sum_{i<j}\frac{1}{|\boldsymbol{x}_i - \boldsymbol{x}_j|} \qquad (2).$$

Since the last term gives a positive contribution to the ground state energy, a lower bound for the "unperturbed" Hamiltonian H_0 (without the mutual Coulomb repulsion) gives a rough lower bound for H_N. But the ground state energy for H_0 is obtained by filling up the Balmer levels. The last completely filled level has principal quantum number n_0 determined by

$$2\sum_{n=1}^{n_0} n^2 \le N \le 2\sum_{n=1}^{n_0+1} n^2 \,,$$

i.e.,

$$\frac{2}{3}n_0(n_0+\frac{1}{2})(n_0+1) \le N \le \frac{2}{3}(n_0+1)(n_0+\frac{3}{2})(n_0+2)\,.$$

The ground state energy E_0 of the unperturbed Hamiltonian satisfies in units of Z^2Ry

$$-\sum_{n=1}^{n_0+1}\frac{2n^2}{n^2} \le E_0 \le -\sum_{n=1}^{n_0}\frac{2n^2}{n^2}\,, \quad [1]$$

i.e. $-2(n_0+1) \le E_0 \le -2n_0$. For large N, $n_0 = (\frac{3N}{2})^{1/3} + O(1)$ the ground state energy E_N of H_N is thus bounded as

$$E_N \ge -2\left(\frac{3}{2}\right)^{1/3} N^{1/3}(1+O(N^{-1/3}))\,Z^2\,Ry \qquad (3).$$

This inequality implies stability of the second kind. For a neutral atom this lower bound is proportional to $Z^{7/3}$.

It is not difficult to derive also an *upper bound* proportional to $N^{1/3}Z^2$, using the variational principle with the Slater determinant belonging to the shell state considered above. Using also the fact that the exchange term is non-positive, as well as the virial theorem for the direct Coulomb term, one easily finds

1 The variable n in the numerator results from the degree of degeneracy of the H-states; the variable n in the denominator as reciprocal of the energies of the H-atom. Both n are identical.

$$E_N \le -2\left(\frac{3}{2}\right)^{1/3}\left(1-\frac{N}{2Z}\right)N^{1/3}\left(1+O(N^{-1/3})\right)Z^2\,Ry \qquad (4).$$

These bounds can be improved.

We note that the *Thomas-Fermi theory* gives for neutral atoms

$$E_N^{TF} = -1.5375\, Z^{7/3}\, Ry \qquad (5).$$

3 Size of large atoms

We are interested in an inequality for the size of an atom in its ground state, defined as

$$r := \left\{ \frac{1}{N}\left\langle \sum_{i=1}^{N} \boldsymbol{x}_i^2 \right\rangle \right\}^{1/2},$$

where the angular bracket denotes the ground state expectation value. One expects, for instance on the basis of the Thomas-Fermi theory, that $r > const\ N^{-1/3}$.

As a first ingredient we use the following operator inequality ($\hbar = 1$):

$$\frac{1}{2}\sum_{i=1}^{N}(\boldsymbol{p}_i^2+\omega^2\boldsymbol{x}_i^2) \ge \omega N^{4/3}\frac{3^{4/3}}{4}\left(1+O(N^{-1/3})\right).$$

This is obtained as the previous inequalities for E_0 of H_0, using that the energy levels of an isotropic harmonic oscillator are $= \frac{3}{2}+n\omega$, with degeneracies $g_n = 2\cdot\frac{1}{2}(n+1)(n+2)$.

Taking now the expectation value of this inequality with the ground state of H_N, and setting

$$\omega = \frac{4}{3^{4/3}N^{4/3}}\left\langle \sum_{i=1}^{N}\boldsymbol{p}_i^2 \right\rangle$$

leads, up to $N^{-1/3}$ corrections, to

$$\left\langle \sum_{i=1}^{N}\boldsymbol{x}_i^2 \right\rangle \ge \frac{(3N)^{8/3}}{16\left\langle \sum_{i=1}^{N}\boldsymbol{p}_i^2 \right\rangle}.$$

Finally, we use the virial theorem for $H_N(\langle T\rangle = |E_N|)$ and the previous lower bound (4) to conclude that in units with $\hbar^2/2m = 1$, $e = 1$:

$$\frac{1}{2}\left\langle \sum_{i=1}^{N}\boldsymbol{p}_i^2 \right\rangle \le \left(\frac{3}{2}\right)^{1/3} N^{7/3}\left(1+O(N^{-1/3})\right).$$

Together this gives

$$\left\langle \sum_{i=1}^{N} x_i^2 \right\rangle \ge \frac{9\cdot 6^{1/3}}{32}N^{1/3},$$

i.e.

$$r \ge 0.71\, N^{-1/3}\left(1+O(N^{-1/3})\right) \qquad (6).$$

Supplementary remarks

For matter in bulk it is not possible to arrive at energy estimates in such an explicit and elementary fashion as for (3) and (4), and one has to use more general methods. As a nice illustration of these, we show how one arrives at a quite accurate lower bound without solving a differential equation, but by making use of the following *Sobolev inequality* [3] for any $\psi \in L^2(\mathbb{R}^3)$ whose (weak) derivatives are also in $L^2(\mathbb{R}^3)$:

$$(\psi, T_e\psi) = \|\nabla\psi\|_2^2 \ge K_s\|\psi\|_6^2 = K_s\|\rho_\psi\|_3 \qquad (7),$$

where $\rho_\psi := |\psi|^2$ and $K_s = 3(\pi/2)^{4/3} \simeq 5.5$ (this numerical value is known to be optimal). This inequality (which is a special case of a whole class), allows us to bound the ground state energy of hydrogen like atoms as

$$E \ge inf_\rho\left\{h(\rho) : \rho(\boldsymbol{x}) \ge 0,\ \int_{\mathbb{R}^3}\rho\, d^3x = 1\right\} \qquad (8),$$

with

$$h(\rho) = K_s\|\rho\|_3 - Z\int\frac{\rho(\boldsymbol{x})}{|\boldsymbol{x}|}\,d^3x \qquad (9).$$

It is straightforward (a nice exercise for students) to find the minimizing ρ, and to show that it gives the lower bound

$$E \ge -\frac{4}{3}Z^2\,Ry\ .$$

This instructive calculation is from Lieb's review paper [4].

One does not loose much by using an even weaker inequality, which has the advantage to be generalizable to many electron systems. This is obtained from (7) with the help of Hölder's inequality:

$$\|fg\|_1 \le \|f\|_p\|g\|_{p'} \qquad \left(\frac{1}{p}+\frac{1}{p'} = 1,\ p \ge 1\right).$$

For $p = 3$, $p' = \frac{3}{2}$ this implies for a normalized ρ

$$\int\rho^{5/3} \le \|\rho\|_3\,\|\rho^{2/3}\|_{3/2} = \left(\int\rho^3\right)^{1/3}\left(\int\rho\right)^{2/3} = \left(\int\rho^3\right)^{1/3}.$$

Hence,

$$(\psi, T_e\psi) \ge K_1\int_{\mathbb{R}^3}\rho_\psi(\boldsymbol{x})^{5/3}\,d^3x \qquad (10)$$

for $K_1 = K_s$, but K_1 can be improved to $K_1 = 9.57$. Instead of (9) we now have the simpler functional

$$h(\rho) = K_1\int\rho^{5/3}\,d^3x - Z\int\frac{\rho}{|\boldsymbol{x}|}\,d^3x \qquad (11),$$

but the lower bound comes out only slightly worse.

For antisymmetric N-electron wave functions, Lieb and Thirring [5] were able to generalize (10), where now ρ_ψ is the one-particle density, normalized as $\int\rho_\psi = N$. With the help of this generalized inequality they were able to bound $(\psi, H_N\psi)$ in terms of the Thomas-Fermi energy functional [2].

4 The Dyson-Lenard-Lieb-Thirring Theorem

I have already mentioned that Dyson and Lenard gave in 1967 a proof of the stability of matter in the sense of (3). This proof was long and involved a large number of estimates. Even in sharp estimations it is unavoidable that about a factor two is lost per page. For a total of 40 pages of the paper one would thus expect a loss of about $2^{40} \sim 10^{14}$, and this is what actually happened. In his preface to Lieb's Selecta [4] Dyson writes:

> *"Our proof was so complicated and so unilluminating that it stimulated Lieb and Thirring to find the first decent proof. Why was our proof so bad and why was theirs so good? The reason is simple. Lenard and I began with ma-*

2 Lieb and Simon [6] showed much earlier that the Thomas-Fermi theory becomes exact in the limit Z → ∞, with the number of nuclei fixed.

thematical tricks and hacked our way through a forest of inequalities without any physical understanding. Lieb and Thirring began with physical understanding and went on to find the appropriate mathematical language to make their understanding rigorous. Our proof was a dead end. Theirs was a gateway to the new world of ideas collected in this book."

Heuristic considerations

On a heuristic level it is easy to understand the stability of the second kind of Coulomb dominated matter. Consider a neutral system of N electrons and N_Z nuclei with charge Ze and mass m_Z ($m_Z \simeq Am_N$, m_N = nucleon mass). Screening effects reduce the effective interactions essentially to one between nearest neighbors. Thus the Coulomb potential energy is roughly (for bosons and fermions)

$$V_{Coul} \approx -N_Z \frac{(Ze)^2}{(R/N_Z^{1/3})},$$

where R is the dimension of the system. For the kinetic energy we have $T \approx Np^2/2m$, where p is the average momentum of the electrons. Roughly speaking, the Pauli principle allows at most one electron in a de Broglie cube $(\hbar/p)^3$, and thus $p \geq N^{1/3}\hbar/R$. For the total energy of the system, we therefore obtain the approximate inequality – including for later purposes also the Newtonian potential energy $-\frac{1}{2}N_Z^2 Gm_Z^2/R$ of the nuclei

$$E \geq N\frac{p^2}{2m} - \frac{1}{2}\left(\frac{N}{Z}\right)^2 \frac{Gm_Z^2}{\hbar N^{1/3}}p - \frac{Ne^2Z^{2/3}}{\hbar}p \qquad (12).$$

The minimum of the right hand side is attained for the average electron momentum p_0, given by

$$Np_0/m = \frac{1}{2}\left(\frac{N}{Z}\right)^2 \frac{Gm_Z^2}{\hbar N^{1/3}} + \frac{Ne^2Z^{2/3}}{\hbar} \qquad (13),$$

in terms of which the ground state energy is $E_0 \approx -Np_0^2/2m$. Ignoring the gravitational interaction, this is *linear* in N:

$$E_0 \approx -N \cdot Ry \qquad (14).$$

For the electron density $n_0 \approx (p_0/\hbar)^3$ and the matter density ρ_0 we obtain, if a_0 denotes the Bohr radius,

$$n_0 \approx Z^2/a_0^3, \qquad \rho_0 \approx AZm_N/a_0^3 \approx 10\ g/cm^3 \qquad (15).$$

If we would treat the electrons as bosons, we would only have the restriction imposed by the uncertainty relation, $p \geq \hbar/R$, and instead of (14) we would obtain

$$E_0 \approx -N^{5/3} \cdot Ry \qquad (bosons) \qquad (16).$$

This $N^{5/3}$ law was established rigorously by Lieb for *fixed positions of the nuclei*. However, when the nuclei are also treated dynamically the $N^{7/5}$ law, discussed earlier, holds.

Rigorous bound

Lieb and Thirring [5] have established the rigorous bound

$$E(N,k) \geq -const \cdot \left\{N + \sum_j^k Z_j^{7/3}\right\} Ry \qquad (17),$$

with a constant of about 20 instead of 10^{14} in the work of Dyson and Lenard. The main step of the proof consists in bounding the ground state energy in terms of the Thomas-Fermi functional. Instead of minimizing this functional, Lieb and Thirring used a theorem of Teller stating that *atoms do not bind in Thomas-Fermi theory* (see [6]). In this way a lower bound in terms of a lower bound of the Thomas-Fermi functional *for atoms* was obtained, for which a previous result of Lieb and Simon could be used.

5 Stability and instability of cold stars

Once gravity becomes important we can no more expect stability of the second kind, because of the purely attractive and long range character of the gravitational interaction. Let us begin with some heuristic considerations.

'Newton begins to dominate Coulomb' when the last two terms in (16) become comparable, i.e., for the 'critical' electron number

$$N_c \approx Z\left(\frac{Z}{A}\right)^3 \alpha^{3/2}\left(\frac{M_{Pl}}{m_N}\right)^3.$$

Here α is the fine structure constant and M_{Pl} the Planck mass. Numerically this is about the number of electrons in Jupiter.
For $N >> N_c$ we can neglect the Coulomb contribution in (12) and then obtain from (13)

$$p_0/mc \approx \frac{1}{2}\left(\frac{A}{Z}\right)^2 \frac{m_N^2}{M_{Pl}^2} N^{2/3}.$$

This shows that the electrons become relativistic for

$$N > N_r := \left(\frac{Z}{A}\right)^3 \left(\frac{2M_{Pl}}{m_N}\right)^{3/2} \qquad (18).$$

Therefore we treat the electrons in (16) relativistically. In units with $c = 1$ we then have

$$E_0(N) \approx inf_p \left\{N\sqrt{p^2+m^2} - \frac{1}{2}\left(\frac{N}{Z}\right)^2 \frac{Gm_Z^2}{\hbar N^{1/3}}\, p\right\} \qquad (19).$$

One readily sees that *the minimum only exists* for $N < N_r$. The corresponding limiting mass

$$M_r = (N_r/Z)\, m_Z \approx 2.8\left(\frac{Z}{A}\right)^2 \frac{M_{Pl}^3}{m_N^2} \qquad (20)$$

is close to the *Chandrasekhar mass*.

The delayed acceptance of the discovery by the 19 year old Chandrasekhar that quantum theory plus special relativity imply the existence of a limiting mass for white dwarfs belongs to the bizarre stories of astrophysics.

The Fowler theory of white dwarfs is just the Thomas-Fermi theory, whereby the white dwarf is considered as a big "atom" with about 10^{57} electrons, and the Chandrasekhar theory is its relativistic version. In other words, the basic Chandrasekhar equation is the same as the relativistic Thomas-Fermi equation (for details see [11]). For white dwarfs the (semi-classical) Thomas-Fermi approximation is ideally justified (a rigorous result will be mentioned below).

In this context the following close historical coincidence is interesting. Thomas' paper [7] was presented at the Cambridge Philosophical Society on November 6, 1926. (Fermi's work was independent, but about one year later.) On the other hand, Fowler communicated his important paper [8] on the non-relativistic theory of white dwarfs about one month later, on December 10, to the Royal Astronomical Society. I wonder who first noticed the close connection of the two approaches.

It is worthwhile mentioning that Lieb and H-T. Yau have shown [9] that Chandrasekhar's theory can be obtained as a limit of a quantum mechanical description in terms of a semi-relativistic Hamiltonian.

In a quantum mechanical description of white dwarfs the proper model to analyze would be a Hamiltonian for electrons and ions with Coulomb and gravitational interactions, for which the kinetic energy of the electrons is the relativistic one:

$$T_e = \sum_{i=1}^{N}\left[\sqrt{\boldsymbol{p}_i^2 + m^2} - m\right] \quad (21).$$

Unfortunately, a rigorous analysis of this model has (to my knowledge) not yet reached a satisfactory stage. A somewhat more modest goal has, however, been reached.

The Coulomb forces in white dwarf material establish *local* neutrality to a very high degree. For this reason the Coulomb interactions play energetically almost no role. (The corrections can be estimated and are on the few percent level.) The spatial distribution of the nuclei and hence their momentum distribution is much the same as those of the electrons. Based on these considerations it is reasonable to expect that the relevant Hamiltonian is

$$H_N = T_e - \sum_{1\le i<j\le N} \frac{G(m_Z/Z)^2}{|\boldsymbol{x}_i - \boldsymbol{x}_j|} \quad (22)$$

($m_Z/Z \simeq (A/Z)\, m_N$ is the mass in the star per electron).

It is now natural to compare the ground state energy

$$E(N) = inf_\psi\, (\psi, H_N\psi)$$

with the semi-classical energy of the Thomas-Fermi theory:

$$E^{TF}(N) = inf\left\{\mathcal{E}^{TF}(n) : \int n = N\right\} \quad (23),$$

where

$$\mathcal{E}^{TF}(n) = \int_{\mathbb{R}^3} \varepsilon(n(\boldsymbol{x}))\, d^3x - \frac{\kappa}{2}\int \frac{n(\boldsymbol{x})\, n(\boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|}\, d^3x d^3x' \quad (24).$$

Here $\kappa = G(m_Z/Z)^2$ and

$$\varepsilon(n) = \frac{1}{\pi^2}\int_0^{p_F(n)} \left[\sqrt{p^2 + m^2} - m\right] p^2\, dp \quad (25),$$

where $p_F(n)$ is the Fermi momentum belonging to the number density n:

$$p_F(n) = (3\pi^2 n)^{1/3} \quad (26).$$

Lieb and H-T. Yau have proved the following

Theorem. *Fix the quantity $\tau = \kappa N^{2/3}$ at some value below the critical value τ_c of the Chandrasekhar theory ($\tau_c \simeq 3.1$). Then*

$$\lim_{N\to\infty} E(N)/E^{TF}(N) = 1 \quad (27).$$

If $\tau > \tau_c$, then

$$\lim_{N\to\infty} E(N) = -\infty \quad (28).$$

As a corollary one can show that the ratio of the critical numbers of electrons for stability becomes 1 in the limit $G \to 0$.

This demonstrates that we can study H_N by means of the semi-classical approximation. This is, of course, not surprising. Indeed, corrections to the Thomas-Fermi approximations are of the order $N^{-1/3}$, i.e., of the order 10^{-19} for $N \sim 10^{57}$. (In contrast to this tiny number for white dwarfs, corrections of the order $Z^{-1/3}$ for atoms are not negligible.)

For an analogous discussion of *boson stars*, I refer again to [9]; see also [10].

6 Concluding remarks

For neutron stars such a quantum mechanical description is not possible, since general relativity has to be used. We are then bound to use a semi-classical description à la Thomas-Fermi, but from what has been said in the last section there can be no doubt that this is an excellent approximation.

When GR is used as the correct theory of gravity, the exclusion principle still influences the magnitudes of limiting masses for stars. But while in Newtonian gravity theory the total energy of a system can be indefinitely negative, this is not the case in GR. The positive energy theorem implies that it is impossible to construct an object out of ordinary matter, whose total energy is negative. (For a detailed proof and discussion, see, e.g., [11]). As a system is compressed to take advantage of the negative gravitational binding energy, a black hole is eventually formed which has positive total energy. This is, however, another story.

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